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THE REGULAR COMPLEX IN THE $BP\langle 1\rangle$–ADAMS SPECTRAL SEQUENCE

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ABSTRACT. We give a complete description of the quotient complex $\mathcal{C}$ obtained by dividing out the $\mathbb{F}_p$ Eilenberg-Mac Lane wedge summands in the first term of the $BP\langle 1\rangle$–Adams spectral sequence for the sphere spectrum S^0. We also give a detailed computation of the cohomology groups $H^{s,t}(\mathcal{C})$ and obtain as a consequence a vanishing line of slope $(p^2 - p - 1)^{-1}$ in their usual $(t - s, s)$ representation. These calculations are interpreted as giving general simple conditions to lift homotopy classes through a $BP\langle 1\rangle$ resolution of S^0.

1. Introduction

One of the main motivations in algebraic topology is the need to have computable methods to analyze homotopical properties of spaces. Indeed, it is often the case that the solution of a given geometrical problem can be reduced to the study of certain key elements in the homotopy groups of a suitable space. The generalized Adams spectral sequence provides a convenient theoretical framework to study these questions; however in practice, even determining whether or not a class is null-homotopic may be an extremely difficult task. This is reflected in the fact that the complexity of calculations with the spectral sequence gets rapidly out of hand. However the analysis of a particular homotopy class can be greatly simplified if the Adams spectral sequence is based at a suitably chosen spectrum, one detecting in a "natural" way the homotopy problem under consideration. Since topological K-theory has shown to be a powerful tool in the solution of geometric problems (Atiyah-Singer Index Theorem or Adams' determination of tangent fields to spheres) it is desirable to have a manageable model to handle calculations in the K-theory Adams spectral sequence. A good deal of homotopical information has already been obtained through calculations directly related to this spectral sequence; for instance this was the approach used to obtain a homotopy theoretical description of the v_1–periodic classes in the stable homotopy groups of spheres in [11], [20] and [21]. Subsequently the techniques have been applied to the study of the stable geometric dimension of vector bundles over projective spaces [9] and also in a geometric construction of the K–theory localization of odd primary Moore spaces [8]. A more recent application of the theory has been made to the classification of the stable p–local homotopy types of stunted spaces arising from the classifying

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space for the group of permutations on p letters (p an odd prime) [14]. The solution of each of the problems above depends on the triviality of certain key homotopy classes or "obstructions", and the success of using the K–theory Adams spectral sequence to analyze them is based in the fact that each time such an obstruction is trivial it vanishes already at the E_2 term of the spectral sequence. Thus while the complete classification of the stable homotopy types of stunted projective spaces [10] was accomplished by showing that many of the relevant obstructions were killed by high–order differentials in the classical Adams spectral sequence, the odd primary problem (stunted lens spaces) was solved in [14] by replacing the analysis of Adams' high–order differentials by the K–theory calculations in that paper.

The second term $E_2(S^0)$ of the Adams spectral sequence based at (a stable summand of connective p–local) K–theory is still far from being completely described, nevertheless the obstructions above can be easily managed within the context of this spectral sequence. The aim of the paper is to give full details on how this can be accomplished. Roughly speaking the analysis is done through a certain quotient complex $\mathcal{C}$ of the first term $E_1(S^0)$ in the spectral sequence, whose cohomology $H^*(\mathcal{C})$ detects the above obstructions. In more detail, letting V stand for the kernel of the projection $E_1(S^0) \to \mathcal{C}$, the usual three-term long exact sequence associated to the extension $0 \to V \to E_1(S^0) \to \mathcal{C} \to 0$ relates $H^*(\mathcal{C})$ to $E_2(S^0)$. The cohomology of $\mathcal{C}$ is computed in full and, together with the long exact sequence just described, produces enough information to show the triviality of the obstructions in $E_2(S^0)$. In a sense $H^*(\mathcal{C})$ is an approximation of $E_2(S^0)$, so that a closer study of $H^*(V)$ would be in order. In this direction we should mention the work in [6] computing V^1, the homological degree $s = 1$ of the complex V, in the 2-primary case. An alternative approach uses a common subcomplex $\mathcal{C}(1)$ of both $\mathcal{C}$ and $E_1(S^0)$. The long exact sequence associated to the extension $0 \to \mathcal{C}(1) \to E_1(S^0) \to E_1(S^0)/\mathcal{C}(1) \to 0$ yields a long exact sequence of the form

$$\cdots \to H^s(\mathcal{C}(1)) \to E_2^s(S^0) \to \mathrm{Ext}^s_{rel}(\mathbb{Z}/p) \to H^{s+1}(\mathcal{C}(1)) \to \cdots,$$

where $\mathrm{Ext}^s_{rel}(\mathbb{Z}/p)$ is a graded group of extensions arising in a certain context of relative homological algebra. $H^*(\mathcal{C}(1))$ is computed in full in this paper, and the relative "Ext" groups are studied in [15].

We now say a few words about the techniques used and the organization of the paper. Section 2 recollects the preliminary results we need and sets up some notation. This section is mainly descriptive, and the results are stated only with references to the literature. We only remark that Theorem 2.7 is a partial generalization of results in [18] and (as noted in that paper) can be proved with similar methods. Moreover the proof of Theorem 2.9 has appeared in print only for the two primary situation ([19]); however the techniques used in that case generalize directly to the odd primary version. The remainder of the paper can be read independently of [19]. Section 3 defines and studies the quotient complex $\mathcal{C}(X)$; in particular a "filtered" homological interpretation of it is derived. Section 4 introduces our main technical tool: Lellmann and Mahowald's weight spectral sequence computing the cohomology of $\mathcal{C}(X)$. The section studies the E_1 term of this spectral sequence, and sections 5 and 6 settle all higher differentials as well as the E_∞ extensions for a number of spectra related to the sphere spectrum. A full description of $H^{s,t}(\mathcal{C}(X))$ follows, as well as a vanishing line of slope $(p^2 - p - 1)^{-1}$ in the $(t - s, s)$ chart for these groups. The calculations are summarized in Corollary 6.11; however, as it stands, it is difficult to appreciate its full strength. Thus the final section derives a topological interpretation (Theorem 7.5) which gives a powerful criterion to lift homotopy classes through a $BP\langle 1\rangle$ resolution of the sphere spectrum. Loosely speaking, this criterion claims that the obstructions to lifting to the second stage in this resolution are given by the image of the J-homomorphism, whereas further obstructions behave almost as in the classical case.

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2. Preliminaries

Unless otherwise specified, all spectra are localized at an odd prime p. We let q, $\mathbb{Z}_{(p)}$ and $\mathbb{F}_p$ stand for $2p-2$, the ring of p–local integers and the field with p elements, respectively. The (p–localization of the) spectrum for connective (real or complex) K–theory decomposes as a wedge of suspensions of a spectrum most commonly denoted by $BP\langle 1\rangle$ ([3], [16]). For simplicity we use Kane's notation ℓ for this spectrum [17]. Thus ℓ is an associative and commutative ring spectrum whose homotopy is a polynomial algebra over $\mathbb{Z}_{(p)}$ on a variable $v \in \pi_q(\ell)$. Let $i : S^0 \to \ell$ denote the unit of ℓ and let $S^0 \xrightarrow{i} \ell \xrightarrow{pr} \overline{\ell}$ be the canonical cofibration. For $s \geq 0$ let $\overline{\ell}^{\,s}$ denote the s–fold smash product of $\overline{\ell}$ with itself and let $S_s = \Sigma^{-s}\overline{\ell}^{\,s}$ ($\overline{\ell}^{\,0} = S^0 = S_0$). The standard ℓ–Adams resolution of a spectrum X is obtained by repeatedly smashing with $i : S^0 \to \ell$ and taking fibers:

$$\begin{array}{ccccccc}
X & \longleftarrow & S_1 \wedge X & \longleftarrow & S_2 \wedge X & \longleftarrow & \cdots \\
\big\downarrow{\scriptstyle i\wedge X} & & \big\downarrow{\scriptstyle i\wedge S_1\wedge X} & & \big\downarrow{\scriptstyle i\wedge S_2\wedge X} & & \\
l \wedge X & & l \wedge S_1 \wedge X & & l \wedge S_2 \wedge X & &
\end{array}$$

The homotopy groups of these fibrations assemble into an exact couple whose associated spectral sequence is the ℓ–Adams spectral sequence for X. The first term of this spectral sequence is given by

$$E_1^{st}(X;\ell) = \pi_{t-s}(\ell \wedge S_s \wedge X) = \pi_t(\ell \wedge \overline{\ell}^{\,s} \wedge X) \tag{1}$$

and the first differential is $\pi_t(\Delta_s)$, where Δ_s is the composite

$$\ell \wedge \overline{\ell}^{\,s} \wedge X \xrightarrow{pr\wedge\overline{\ell}^{\,s}\wedge X} \overline{\ell} \wedge \overline{\ell}^{\,s} \wedge X = S^0 \wedge \overline{\ell}^{\,s+1} \wedge X \xrightarrow{i\wedge\overline{\ell}^{\,s+1}\wedge X} \ell \wedge \overline{\ell}^{\,s+1} \wedge X. \tag{2}$$

Definition 2.1. Margolis showed [23] that a connective, locally finite (p–local) spectrum X can be decomposed in a unique way, up to nonnatural equivalences, as a wedge sum $X \simeq KV(X) \vee X^{\langle 0\rangle}$, where $V(X)$ is a locally finite, graded $\mathbb{F}_p$ vector space and $KV(X)$ is the associated wedge of $\mathbb{F}_p$ Eilenberg-Mac Lane spectra, and where $X^{\langle 0\rangle}$ has no further such summands. We also define spectra $X^{\langle i\rangle}$ ($i > 0$) and maps $X^{\langle i\rangle} \to X^{\langle i-1\rangle}$ (Adams projections) by requiring that $\cdots \to X^{\langle 2\rangle} \to X^{\langle 1\rangle} \to X^{\langle 0\rangle}$ be a minimal $\mathbb{F}_p$–Adams resolution of $X^{\langle 0\rangle}$. The homotopy types of these spectra and maps are again unique up to nonnatural equivalences [1, pg. 28]. Due to the lack of functoriality, choices will always be present in handling these constructions.

Definition 2.2. For $s \geq 1$ let R_s denote the set of s–tuples of positive integers. We will generally denote an element $(n_1, \cdots, n_s) \in R_s$ simply by $\overline{n}$, and define $\sigma(\overline{n}) = \sum n_i$ and $\nu(\overline{n}!) = \sum \nu(n_i!)$, where $\nu(a)$ denotes the maximal power of p dividing the integer a.

The analysis of the groups in (1) was began by Mahowald for the two primary case. The odd primary version of his result is given by the following theorem.

Theorem 2.3 ([8], [17], [18])**.** *Let $s \geq 1$. For $\overline{n} \in R_s$ there is a spectrum $B_{\overline{n}}$ and an equivalence $(\ell \wedge B_{\overline{n}})^{\langle 0 \rangle} \simeq \Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!) \rangle}$. Moreover $\ell \wedge \overline{\ell}^{\,s}$ is homotopy equivalent to $\bigvee_{\overline{n} \in R_s} \ell \wedge B_{\overline{n}}$.*

In terms of these splittings the maps in (2) have components $\ell \wedge B_{\overline{n}} \to \ell \wedge B_{\overline{m}}$ (for $X = S^0$) which neglecting the Eilenberg-Mac Lane summands (in a way to be made precise in the next section) are operations of the form $\Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!) \rangle} \to \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!) \rangle}$. Operations of this sort were studied by Lellmann, and we now describe his results.

Definition 2.4. Let $b \in \mathbb{Z}$ be a representative of a generator of the units in $\mathbb{Z}/p^2$, and consider the stable Adams operation $\Psi^b : \ell \to \ell$. According to [18] the operation $\Psi^b - 1$ lifts in a unique way through $v : \Sigma^q \ell \to \ell$—the ℓ-modulification of $v \in \pi_q(\ell)$—defining an operation $\phi : \ell \to \Sigma^q \ell$ whose behavior in homotopy is given by

$$\phi(v^r) = (k^r - 1)v^{r-1}, \quad \text{where} \quad k = b^{p-1}. \tag{3}$$

Definition 2.5 ([13]). For nonnegative integers $m \leq n$, define the Gaussian coefficient $(\!(\begin{smallmatrix} n \\ m \end{smallmatrix})\!)$ and the Gaussian factorial $n!!$ by

$$(\!(\begin{smallmatrix} n \\ m \end{smallmatrix})\!) = \frac{(k^n - 1)(k^{n-1} - 1) \cdots (k^{n-m+1} - 1)}{(k^m - 1)(k^{m-1} - 1) \cdots (k - 1)},$$

$$n!! = (\!(\begin{smallmatrix} n \\ 1 \end{smallmatrix})\!)(\!(\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix})\!) \cdots (\!(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})\!),$$

where k is given as in (3). The formula $\nu(k^n - 1) = \nu(n) + 1$ [2, Lemma 2.12] implies that these numbers lie in $\mathbb{Z}_{(p)}$, and also give us the relations $\nu(n!!) = \nu(n!)$ and $\nu(\!(\begin{smallmatrix} n \\ m \end{smallmatrix})\!) = \nu(\begin{smallmatrix} n \\ m \end{smallmatrix})$.

Definition 2.6. For a nonnegative integer n let $\Pi_n : \ell^{\langle \nu(n!) \rangle} \to \ell$ be the composite of the Adams projections $\ell^{\langle \nu(n!) \rangle} \to \ell^{\langle \nu(n!)-1 \rangle} \to \cdots \to \ell^{\langle 0 \rangle} = \ell$ and the degree r map in ℓ, where $r = n!!/p^{\nu(n!)}$, which is a unit in $\mathbb{Z}_{(p)}$. Lellmann showed that the $n^{\text{ th}}$ power ϕ^n of ϕ lifts through Π_n, defining an operation

$$\phi_n : \ell \to \Sigma^{nq} \ell^{\langle \nu(n!) \rangle}. \tag{4}$$

The effect in homotopy groups of each ϕ_n follows directly from (3): $\phi_n(v^r) = (\!(\begin{smallmatrix} r \\ n \end{smallmatrix})\!)(k-1)^n v^{r-n}$. Since $\pi_*(\ell)$ is torsion free, it follows that no ϕ_n is a torsion operation. Such operations are quite manageable in view of the following result, which is a generalization of results in [18].

Theorem 2.7. *Operations of the form*

$$\Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!) \rangle} \to \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!) \rangle} \tag{5}$$

are uniquely determined up to torsion operations by their effect in homotopy groups. Moreover, torsion operations of the form (5) *factor as*

$$\Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!) \rangle} \xrightarrow{\alpha} KV \xrightarrow{\beta} \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!) \rangle},$$

where V is a locally finite, graded $\mathbb{F}_p$ vector space and KV is the associated wedge of $\mathbb{F}_p$ Eilenberg-Mac Lane spectra.

Using the techniques in [19] one can compute enough of the effect in homotopy groups of the maps Δ_s in (2) (for the case $X = S^0$), so that 2.3 and 2.7 give a description of the components for Δ_s (Theorem 2.9 below), up to operations factoring through $\mathbb{F}_p$ Eilenberg-Mac Lane spectra.

Definition 2.8. Let $\overline{n} = (n_1, \dots, n_s) \in R_s$. A successor of $\overline{n}$ is any $(s+1)$-tuple $\overline{m} \in R_{s+1}$ of the form $\overline{m} = (n_1, \dots, n_{e-1}, j, n_e - j, n_{e+1}, \dots, n_s)$ for some $1 \le e \le s$ and $0 < j < n_e$. Note that $\nu(\overline{n}!) = \nu(\overline{m}!) + \nu\binom{n_e}{j}$. In this case, for a spectrum X we define the degree $(\!(\binom{n_e}{j})\!)$ Adams projection $\Pi_{\overline{n},\overline{m}} : (\ell \wedge X)^{\langle \nu(\overline{n}!)\rangle} \to (\ell \wedge X)^{\langle \nu(\overline{m}!)\rangle}$ to be the composite of the $\nu\binom{n_e}{j}$ Adams projections $(\ell \wedge X)^{\langle \nu(\overline{n}!)\rangle} \to (\ell \wedge X)^{\langle \nu(\overline{n}!)-1\rangle} \to \cdots \to (\ell \wedge X)^{\langle \nu(\overline{m}!)\rangle}$ and the degree r map in $(\ell \wedge X)^{\langle \nu(\overline{m}!)\rangle}$, where $r = (\!(\binom{n_e}{j})\!) p^{-\nu\binom{n_e}{j}}$.

Theorem 2.9 ([19], $p = 2$)**.** *Let $\overline{n} \in R_s$, $\overline{m} \in R_{s+1}$, and let $\ell \wedge B_{\overline{n}} \xrightarrow{\varphi} \ell \wedge B_{\overline{m}}$ be the corresponding component (Theorem 2.3) for the map $\Delta_s : \ell \wedge \overline{\ell}^{\,s} \to \ell \wedge \overline{\ell}^{\,s+1}$ given in (2) for the case $X = S^0$. Let i and π denote the wedge inclusion and projection $\Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!)\rangle} \hookrightarrow \ell \wedge B_{\overline{n}}$ and $\ell \wedge B_{\overline{m}} \to \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!)\rangle}$ respectively. Then there exist a spectrum K which is a locally finite wedge of $\mathbb{F}_p$ Eilenberg-Mac Lane spectra and maps*

$$\alpha : \Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!)\rangle} \to K \quad \textit{and} \quad \beta : K \to \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!)\rangle}$$

such that one of the options below holds for the difference

$$D_{\overline{n},\overline{m}} = \pi\varphi i - \beta\alpha : \Sigma^{\sigma(\overline{n})q} \ell^{\langle \nu(\overline{n}!)\rangle} \to \Sigma^{\sigma(\overline{m})q} \ell^{\langle \nu(\overline{m}!)\rangle}.$$

a) *It agrees with $(-1)^e \Sigma^{\sigma(\overline{n})q} \Pi_{\overline{n},\overline{m}}$ if $\overline{m}$ is a successor of $\overline{n}$ where e and $\Pi_{\overline{n},\overline{m}}$ are as in 2.8 (taking $X = S^0$).*
b) *It is compatible through Adams projections with the $\sigma(\overline{n})q$ suspension of ϕ_{m_1} : $\ell \to \Sigma^{m_1 q} \ell^{\langle \nu(m_1!)\rangle}$ if $m_{r+1} = n_r$ for $r = 1, \dots, s$.*
c) *It is trivial in any other case.*

Remark 2.10. For $s = 0$ and $m = (1)$, the component $\ell \to \Sigma^q \ell$ in 2.9 is in fact ϕ (that is, $\beta \circ \alpha$ is trivial) in view of [18, Theorem 2.2 (iv)].

3. The ℓ–regular cochain complex

We now extend the description of Δ_s given in Theorem 2.9 above ($X = S^0$) to a more general type of spectra X. From now on $H\mathbb{F}_p$ will denote the $\mathbb{F}_p$ Eilenberg-Mac Lane spectrum.

Definition 3.1. A (p–local) connective, locally finite spectrum X is said to be $(\ell, H\mathbb{F}_p)$–prime ([19], [26]) if the $\mathbb{F}_p$–Adams spectral sequence for $\ell \wedge X$ converges to $\ell_*(X)$ and collapses from its E_2 term. We will require in addition that the composite $(\Sigma^q \ell \wedge X)^{\langle 0\rangle} \hookrightarrow \Sigma^q \ell \wedge X \to \ell \wedge X \to (\ell \wedge X)^{\langle 0\rangle}$ yields a monomorphism in homotopy groups, where the first and third maps are the wedge inclusion and projection respectively, and the middle one is induced by the map $v : \Sigma^q \ell \to \ell$ in 2.4.

Remark 3.2. When X is an $(\ell, H\mathbb{F}_p)$–prime spectrum, all Adams projections $(\ell \wedge X)^{\langle i+1\rangle} \to (\ell \wedge X)^{\langle i\rangle}$ induce monomorphisms in homotopy groups, for a nontrivial homotopy class in $(\ell \wedge X)^{\langle i+1\rangle}$ mapping trivially into $(\ell \wedge X)^{\langle i\rangle}$ would produce by definition a nontrivial differential d_r in the $\mathbb{F}_p$–Adams spectral sequence for $\ell \wedge X$ (r would be at least two, since d_1 differentials are trivial when minimal resolutions are used). In particular the last condition in 3.1, together with the considerations in 3.4 below, implies that $X^{\langle i \rangle}$ is $(\ell, H\mathbb{F}_p)$–prime whenever X is. This extra condition allows an elementary proof of Lemma 3.3 as well as a conceptually cleaner interpretation (Proposition 3.10) of the ℓ regular complex $\mathcal{C}(X)$ (to be defined in 3.5). Spheres and stunted lens spaces are examples of $(\ell, H\mathbb{F}_p)$–prime spectra.

Throughout the rest of the paper X will always denote an $(\ell, H\mathbb{F}_p)$–prime spectrum. Such spectra have the following property.

Lemma 3.3. *Let K be a locally finite wedge of $\mathbb{F}_p$ Eilenberg-Mac Lane spectra. Then any map $\Sigma^r K \stackrel{\alpha}{\to} (\ell \wedge X)^{\langle i \rangle}$ induces trivial homomorphisms in homotopy groups.*

Proof. We can assume $K = H\mathbb{F}_p$. Since the map $v : \Sigma^q \ell \to \ell$ in 2.4 is of positive classical Adams filtration, Margolis' theorem [23, Theorem 3] implies that the composite $\Sigma^r K \stackrel{\alpha}{\to} (\ell \wedge X)^{\langle i \rangle} \to (\ell \wedge X)^{\langle 0 \rangle} \to (\Sigma^{-q}\ell \wedge X)^{\langle 0 \rangle}$ is null homotopic, where the middle map is the Adams projection and the last one is a suitable suspension of that in 3.1. The result follows from the hypothesis that the last two maps in this composite are injective in homotopy groups. ∎

Remark 3.4. The wedge inclusion $(\ell \wedge X)^{\langle 0 \rangle} \stackrel{j}{\hookrightarrow} \ell \wedge X$ induces a map from the minimal $\mathbb{F}_p$–Adams resolution of $(\ell \wedge X)^{\langle 0 \rangle}$ into the Adams resolution $\cdots \to \ell^{\langle 1 \rangle} \wedge X \to \ell^{\langle 0 \rangle} \wedge X$. This map of resolutions can be chosen so that each resulting map $(\ell \wedge X)^{\langle i \rangle} \hookrightarrow \ell^{\langle i \rangle} \wedge X$ agrees with the wedge inclusion of $(\ell^{\langle i \rangle} \wedge X)^{\langle 0 \rangle}$ into $\ell^{\langle i \rangle} \wedge X$. Similarly the wedge projection $\ell \wedge X \stackrel{\pi}{\to} (\ell \wedge X)^{\langle 0 \rangle}$ induces a corresponding map between the above resolutions; moreover choices can be made so that each resulting map $\ell^{\langle i \rangle} \wedge X \to (\ell \wedge X)^{\langle i \rangle}$ agrees with the wedge projection of $\ell^{\langle i \rangle} \wedge X$ onto $(\ell^{\langle i \rangle} \wedge X)^{\langle 0 \rangle}$ and so that each composite $(\ell \wedge X)^{\langle i \rangle} \hookrightarrow \ell^{\langle i \rangle} \wedge X \to (\ell \wedge X)^{\langle i \rangle}$ is the identity. Thus in particular $(\ell^{\langle i \rangle} \wedge X)^{\langle 0 \rangle} = (\ell \wedge X)^{\langle i \rangle}$. Likewise $(\ell \wedge X^{\langle i \rangle})^{\langle 0 \rangle} = (\ell \wedge X)^{\langle i \rangle}$.

To simplify notation we write Margolis' decomposition 2.1 for the spectrum $\ell \wedge \overline{\ell}^{\,s} \wedge X$ as $\ell \wedge \overline{\ell}^{\,s} \wedge X \simeq KV^s(X) \vee Y^s(X)$, so that $Y^s(X) = (\ell \wedge \overline{\ell}^{\,s} \wedge X)^{\langle 0 \rangle}$ and $KV^s(X)$ is the locally finite wedge of $\mathbb{F}_p$ Eilenberg-Mac Lane spectra associated to the graded $\mathbb{F}_p$ vector space $V^s(X)$. Since any smash product $KV \wedge X$ is again a KV–type spectrum [24, Proposition 6.6], it follows from 2.3 and 3.4 that

$$Y^s(X) \simeq \bigvee_{\overline{n} \in R_s} \Sigma^{\sigma(\overline{n})q} (\ell \wedge X)^{\langle \nu(\overline{n}!) \rangle}, \tag{6}$$

and equation (1) gives

$$E_1^{st}(X; \ell) = \pi_t(Y^s(X)) \oplus V^{st}(X); \tag{7}$$

moreover the differential in $E_1(X;\ell)$, as given in (2), is invariant on $V(X)$ in view of 3.3 and (6).

Definition 3.5. The ℓ–regular cochain complex $\mathcal{C}(X)$ is defined to be the quotient complex of $E_1(X;\ell)$ by the subcomplex $V(X)$.

Thus $\mathcal{C}^{st}(X) = \pi_t(Y^s(X))$, and the differential in $\mathcal{C}(X)$ is induced by the composite
$Y^s(X) \stackrel{j}{\hookrightarrow} \ell \wedge \overline{\ell}^{\,s} \wedge X \xrightarrow{\Delta_s} \ell \wedge \overline{\ell}^{\,s+1} \wedge X \stackrel{\pi}{\to} Y^{s+1}(X)$, where j and π are the wedge

inclusion and projection respectively and Δ_s is given in (2). By (6) this differential has components of the form

$$\pi_*\left(\Sigma^{\sigma(\overline{n})q}(\ell\wedge X)^{\langle\nu(\overline{n}!)\rangle}\right)\longrightarrow \pi_*\left(\Sigma^{\sigma(\overline{m})q}(\ell\wedge X)^{\langle\nu(\overline{m}!)\rangle}\right) \tag{8}$$

for $\overline{n}\in R_s$ and $\overline{m}\in R_{s+1}$, each of which is induced by the corresponding composite

$$\begin{aligned}\Sigma^{\sigma(\overline{n})q}(\ell\wedge X)^{\langle\nu(\overline{n}!)\rangle} &\overset{j}{\hookrightarrow} \Sigma^{\sigma(\overline{n})q}\ell^{\langle\nu(\overline{n}!)\rangle}\wedge X \overset{j}{\hookrightarrow} \ell\wedge B_{\overline{n}}\wedge X\\ &\overset{j}{\hookrightarrow} \ell\wedge\overline{\ell}^{\,s}\wedge X \xrightarrow{\Delta_s} \ell\wedge\overline{\ell}^{\,s+1}\wedge X\\ &\xrightarrow{\pi} \ell\wedge B_{\overline{m}}\wedge X \xrightarrow{\pi} \Sigma^{\sigma(\overline{m})q}\ell^{\langle\nu(\overline{m}!)\rangle}\wedge X\\ &\xrightarrow{\pi} \Sigma^{\sigma(\overline{m})q}(\ell\wedge X)^{\langle\nu(\overline{m}!)\rangle},\end{aligned} \tag{9}$$

where each map j and π is either a wedge inclusion or projection given by 2.3 or 3.4.

Theorem 3.6. a) *With the notation of 2.8, if $\overline{m}$ is a successor of $\overline{n}$, (8) is induced by $(-1)^e\Sigma^{\sigma(\overline{n})q}\Pi_{\overline{n},\overline{m}}$.*

b) *If $m_{r+1}=n_r$ for $r=1,\dots,s$, (8) is induced by a map compatible through Adams projections with the $\sigma(\overline{n})q$–suspension of the composite*

$$(\ell\wedge X)^{\langle 0\rangle}\hookrightarrow \ell\wedge X\xrightarrow{\phi_{m_1}\wedge X}\Sigma^{m_1q}\ell^{\langle\nu(m_1!)\rangle}\wedge X\to\Sigma^{m_1q}(\ell\wedge X)^{\langle\nu(m_1!)\rangle},$$

where the first and last maps are the wedge inclusion and projection respectively.

c) *In any other case (8) is trivial.*

Proof. In the notation of 2.9 we see that the composite in (9) can be rewritten as

$$\begin{aligned}\Sigma^{\sigma(\overline{n})q}(\ell\wedge X)^{\langle\nu(\overline{n}!)\rangle} &\overset{j}{\hookrightarrow}\Sigma^{\sigma(\overline{n})q}\ell^{\langle\nu(\overline{n}!)\rangle}\wedge X\xrightarrow{(\pi\varphi i)\wedge X}\Sigma^{\sigma(\overline{m})q}\ell^{\langle\nu(\overline{m}!)\rangle}\wedge X\\ &\xrightarrow{\pi}\Sigma^{\sigma(\overline{m})q}(\ell\wedge X)^{\langle\nu(\overline{m}!)\rangle},\end{aligned}$$

which by 3.3 induces in homotopy the same morphism as the composite obtained by replacing the middle map above by $D_{\overline{n},\overline{m}}\wedge X$. The result follows from 2.9 and the considerations in 3.4. □

Definition 3.7. Let R be a commutative ring with a decreasing filtration $F^iR\supseteq F^{i+1}R$ satisfying $F^iR\cdot F^jR\subseteq F^{i+j}R$. We consider filtered graded R–modules M and M' with $F^iM\supseteq F^{i+1}M$ and $F^iR\cdot F^jM\subseteq F^{i+j}M$. A homomorphism $f:M\to M'$ is said to be filtration preserving if $f(F^iM)\subseteq F^iM'$. The tensor product filtration makes $M\otimes M'$ a filtered R module (see for instance (20) in the next section). A filtered graded coalgebra over R is a graded coalgebra Γ which is also a filtered graded R module in such a way that both structural maps $\varepsilon:\Gamma\to R$ and $\Delta:\Gamma\to\Gamma\otimes\Gamma$ are filtration preserving. A filtered graded right comodule over Γ is a graded right comodule M over Γ which is also a filtered graded R module in such a way that the structural map $M\overset{\varphi}{\to}M\otimes\Gamma$ is filtration preserving. Filtered graded left comodules are defined analogously.

Example 3.8. On the ring $\mathbb{Z}_{(p)}$ consider the (nonnegative) filtration defined by $F^i\mathbb{Z}_{(p)}=\{x\in\mathbb{Z}_{(p)}|\nu(x)\geq i\}$ ($\nu(x)=\nu(a)-\nu(b)$ if $x=\frac{a}{b}$, $a,b\in\mathbb{Z}$, where $\nu(a)$ and $\nu(b)$ are as in 2.2). Define a filtered graded coalgebra Γ over $\mathbb{Z}_{(p)}$ by $\Gamma^k=0$ if $k\not\equiv 0\,(\mathrm{mod}\ q)$ and Γ^{jq} is a $\mathbb{Z}_{(p)}$–free module with generator t_j of filtration

$-\nu(j!)$. The coproduct Δ is defined by $\Delta(t_i) = \sum_{r+s=i} t_r \otimes t_s$ and the counit by $\varepsilon(t_0) = 1$. For any spectrum X, $\ell_*(X)$ becomes a filtered graded right comodule over Γ with coaction $\Psi : \ell_*(X) \to \ell_*(X) \otimes \Gamma$ given by $\Psi(z) = \sum_{n\geq 0} \phi^n(z) \otimes t_n$, where $\phi : \ell \to \Sigma^q \ell$ is defined in 2.4 and where the filtration in $\ell_*(X)$ is the usual $\mathbb{F}_p$–Adams filtration (Ψ is filtration preserving in view of 2.6). If in addition X is an $(\ell, H\mathbb{F}_p)$–prime spectrum, then $\pi_*((\ell\wedge X)^{\langle 0\rangle})$ is also a filtered graded right comodule over Γ. In fact, by 3.3 the wedge projection $\ell \wedge X \to (\ell\wedge X)^{\langle 0\rangle}$ yields an epimorphism $\ell_*(X) \to \pi_*((\ell\wedge X)^{\langle 0\rangle})$ of filtered graded right comodules. In the quotient, the action of ϕ^n on an element $z \in \pi_*((\ell\wedge X)^{\langle 0\rangle})$ will still be denoted by $\phi^n(z)$. In particular, the Γ coaction on $\pi_*((\ell\wedge X)^{\langle 0\rangle})$ is described by the same formula as the corresponding one on $\ell_*(X)$.

Note 3.9. Let Γ be as above. For filtered graded right and left Γ comodules M and N respectively, let $C_\Gamma(M,N)$ denote the cobar complex [28, A1.2.11]. Taking the tensor product filtration on each $C^s_\Gamma(M,N) = M \otimes \overline{\Gamma}^{\otimes s} \otimes N$, we obtain a corresponding filtration on the complex $C_\Gamma(M,N)$, which we call the Adams filtration on $C_\Gamma(M,N)$. For $i \in \mathbb{Z}$ let $AF^iC_\Gamma(M,N)$ be the subcomplex consisting of elements in $C_\Gamma(M,N)$ having Adams filtration at least i. As shorthand, we will denote by $AF^i(X)$ the complex $AF^iC_\Gamma(\pi_*((\ell\wedge X)^{\langle 0\rangle}), \mathbb{Z}_{(p)})$.

Theorem 3.10 ([19], $p = 2$)**.** *There is an isomorphism $\mathcal{C}(X) \simeq AF^0(X)$ of filtered complexes.*

Proof. For $\overline{n} \in R_s$ let $\Pi_{\overline{n}} : (\ell\wedge X)^{\langle\nu(\overline{n}!)\rangle} \to (\ell\wedge X)^{\langle 0\rangle}$ be the composite of the Adams projections $(\ell\wedge X)^{\langle\nu(\overline{n}!)\rangle} \to (\ell\wedge X)^{\langle\nu(\overline{n}!)-1\rangle} \to \cdots \to (\ell\wedge X)^{\langle 0\rangle}$ and the degree r map in $(\ell\wedge X)^{\langle 0\rangle}$, where $r = n_1!!\cdots n_s!!\, p^{-\nu(\overline{n}!)}$ (see 2.8). By 3.2 and (6) these maps yield isomorphisms

$$\mathcal{C}^{st}(X) = \bigoplus_{\overline{n}\in R_s} \pi^{\nu(\overline{n}!)}_{t-\sigma(\overline{n})q}(\ell\wedge X), \tag{10}$$

where $\pi^f_n(\ell\wedge X)$ stands for the subgroup of $\pi_n(\ell\wedge X)$ consisting of the homotopy classes of classical Adams filtration at least f (when $f = 0$, $\pi^f_n(\ell\wedge X)$ should be interpreted as $\pi_n((\ell\wedge X)^{\langle 0\rangle})$. In view of 2.6 and 3.6 the differential $\mathcal{C}^{s,t}(X) \to \mathcal{C}^{s+1,t}(X)$ extends to a map

$$\bigoplus_{\overline{n}\in R_s} \pi_{t-\sigma(\overline{n})q}((\ell\wedge X)^{\langle 0\rangle}) \longrightarrow \bigoplus_{\overline{m}\in R_{s+1}} \pi_{t-\sigma(\overline{m})q}((\ell\wedge X)^{\langle 0\rangle}) \tag{11}$$

whose component $\pi_{t-\sigma(\overline{n})q}((\ell\wedge X)^{\langle 0\rangle}) \to \pi_{t-\sigma(\overline{m})q}((\ell\wedge X)^{\langle 0\rangle})$ is trivial, unless:

a) $\overline{m}$ is a successor of $\overline{n}$, in which case it is multiplication by $(-1)^e$ where e is as in 2.8; or
b) $m_{r+1} = n_r$, $r = 1,\ldots,s$, in which case it is induced by the composite $(\ell\wedge X)^{\langle 0\rangle} \hookrightarrow \ell\wedge X \to \Sigma^{m_1q}\ell\wedge X \to (\Sigma^{m_1q}\ell\wedge X)^{\langle 0\rangle}$, where the first and third maps are the wedge inclusion and projection respectively, and the middle one is induced by $\phi^{m_1} : \ell \to \Sigma^{m_1q}\ell$.

Identification of an element $z \in \pi_{t-\sigma(\overline{n})q}((\ell\wedge X)^{\langle 0\rangle})$ with the corresponding

$$z[t_{n_1}|\cdots|t_{n_s}] \in C^{s,t}_\Gamma(\pi_*((\ell\wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$$

(where $\overline{n} = (n_1,\ldots,n_s)$) sets an isomorphism

$$h : \bigoplus_{\overline{n}\in R_s} \pi_{t-\sigma(\overline{n})q}((\ell\wedge X)^{\langle 0\rangle}) \simeq C^{s,t}_\Gamma(\pi_*((\ell\wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$$

which restricts to an isomorphism $\mathcal{C}(X) \simeq AF^0(X)$. The result follows since, by definition, the differential in $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ corresponds under h to the description above for the map in (11) (see (12) in the next section or, directly, [28, A1.2.11]). □

4. The weight spectral sequence

Throughout the rest of the paper Γ will be the filtered coalgebra defined in 3.8. In the last section the regular complex $\mathcal{C}(X)$ is identified with the subcomplex of $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}), \mathbb{Z}_{(p)})$ consisting of the elements of nonnegative Adams filtration. In particular, $\mathcal{C}(X)$ is a (nonnegative) filtered cochain complex and its cohomology groups could be analyzed via the spectral sequence associated to this filtration. This approach is effective when the Γ coaction on $\pi_*((\ell \wedge X)^{\langle 0\rangle})$ is trivial (which in general is not the case). The analysis of the (2–primary) general situation was completed in [19] by considering a different filtration on $\mathcal{C}(X)$. In this section we give a description of the first term of the resulting spectral sequence in the odd primary case. We begin by fixing some notation. For a filtered graded Γ–comodule M ($M \neq \mathbb{Z}_{(p)}$) we denote by d the differential in $C_\Gamma(M; \mathbb{Z}_{(p)})$. Only the differential in $C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ will be denoted by ∂. When $M = \pi_*((\ell \wedge X)^{\langle 0\rangle})$, the restriction of d to $\mathcal{C}(X)$ will be denoted by d too. A cobar product $[t_{n_1}|\cdots|t_{n_s}]$ in $C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ will be denoted by $[t_{\overline{n}}]$, where $\overline{n} = (n_1, \ldots, n_s)$.

Definition 4.1. For a filtered graded Γ–comodule M we define the total degree, the homological degree and the weight filtration degree of an element $m[t_{\overline{n}}] \in C^s_\Gamma(M; \mathbb{Z}_{(p)})$ as $r + q\sigma(\overline{n})$, s and $\sigma(\overline{n})$ respectively, where $m \in M_r$ and $\sigma(\overline{n})$ is as in 2.2. The weight filtration degree of $m[t_{\overline{n}}]$ will be denoted by $WF(m[t_{\overline{n}}])$.

As usual the differential in $C_\Gamma(M; \mathbb{Z}_{(p)})$ depends both on the coalgebra Γ (that is, ∂) and the Γ coaction on M (see for instance [28, A1.2.11]). In particular, when $M = \pi_*((\ell \wedge X)^{\langle 0\rangle})$, 3.8 gives

$$\begin{aligned} \partial[t_{\overline{n}}] &= \sum_{\overline{m}} (-1)^{e(\overline{m})}[t_{\overline{m}}], \\ d(z[t_{\overline{n}}]) &= \sum_{n_0 \geq 1} \phi^{n_0}(z)[t_{n_0}|t_{\overline{n}}] + z\partial[t_{\overline{n}}], \end{aligned} \tag{12}$$

where the summation on the right hand side of the first formula runs over the set of all successors $\overline{m} \in R_{s+1}$ of $\overline{n}$ with $\overline{m} = (n_1, \ldots, n_{e(\overline{m})-1}, j(\overline{m}), n_{e(\overline{m})} - j(\overline{m}), n_{e(\overline{m})+1}, \ldots, n_s)$, $1 \leq e(\overline{m}) \leq s$ and $0 < j(\overline{m}) < n_{e(\overline{m})}$ (see 2.8). For a nonnegative integer σ, (12) implies that the elements of weight filtration at least σ in $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}), \mathbb{Z}_{(p)})$ generate a subcomplex $WF^\sigma(X)$, and this produces a (nonnegative) decreasing filtration on $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}), \mathbb{Z}_{(p)})$ which we call the weight filtration. By restriction to $\mathcal{C}(X)$ we obtain a corresponding weight filtration on $\mathcal{C}(X)$.

Definition 4.2. The spectral sequence associated to the weight filtration on $\mathcal{C}(X)$ is called the weight spectral sequence (WSS) for X and is denoted by $\{E_r(X), \delta_r\}$.

Standard references for spectral sequences arising from filtered complexes are [5] and [12]; we remark however that what is called the total degree in [5] corresponds to our homological degree. In 5.5 of the next section we give a quick example on how differentials are computed in these spectral sequences. Our indexing for the

WSS is that of [12]:

$$E_0^{\sigma st}(X) = \frac{WF^{\sigma st}}{WF^{\sigma+1,s,t}} \ , \tag{13}$$

where $WF^{\sigma st} = WF^{\sigma}(X) \cap \mathcal{C}^{st}(X)$. Thus elements in $E_r^{\sigma st}(X)$ have representatives $x \in WF^{\sigma st}$ whose differential dx lands in $WF^{\sigma+r,s+1,t}$, so that the grading for the differentials is

$$\delta_r : E_r^{\sigma st}(X) \to E_r^{\sigma+r,s+1,t}(X). \tag{14}$$

As in 4.1, the indices σ, s, t are called the weight filtration degree, the homological degree and the total degree respectively.

Remark 4.3. The total degree in $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ does not change under the differential d, and therefore the WSS for X is actually a set of spectral sequences, one for each total degree $t \geq 0$. Moreover an element $z[t_{\overline{n}}] \in WF^{\sigma st}$ of weight filtration degree $\sigma \gg t/q$ must have "coefficient" $z \in \pi_r((\ell \wedge X)^{\langle 0\rangle})$ with $r \ll 0$ (see 4.1). Since X is a connective spectrum (in the sense that $\pi_*(X) = 0$ for $*$ small enough) we see that at a given total degree t the weight filtration on $\mathcal{C}(X)$ is bounded (uniformly on the homological degree s) and therefore the corresponding spectral sequence collapses at a finite stage (which depends on the total degree taken). In particular, the WSS for X does converge to the cohomology groups of $\mathcal{C}(X)$.

In view of (10) the subcomplex $WF^{\sigma+1}$ in (13) is a direct summand of WF^{σ}. This yields an obvious trigraded isomorphism $\mathcal{C}(X) \simeq E_0(X)$ (not of complexes), where the extra grading in $\mathcal{C}(X)$ corresponding to the weight filtration degree. Moreover, in view of (12) the differential in $E_0(X)$ differs from that in $\mathcal{C}(X)$ only by the Γ–coaction on $\pi_*((\ell \wedge X)^{\langle 0\rangle})$—in $E_0(X)$ this coaction is not present. These remarks apply also to $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ instead of $\mathcal{C}(X)$ to produce the trigraded isomorphism

$$C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)}) \simeq E_0 C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)}). \tag{15}$$

Again, this is not an isomorphism of complexes; however the right hand side in (15)—that is, the quotient object associated to the weight filtration on the cobar complex $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$— is isomorphic as a complex to

$$\pi_*((\ell \wedge X)^{\langle 0\rangle}) \otimes C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}).$$

Since the above isomorphisms preserve Adams filtration, 3.10 gives

Proposition 4.4. *As a trigraded cochain complex, $E_0(X)$ is isomorphic to the subcomplex of $\pi_*((\ell \wedge X)^{\langle 0\rangle}) \otimes C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ consisting of elements of nonnegative Adams filtration.*

Definition 4.5. The spectral sequences associated to the Adams filtration on the complexes $E_0(X)$ and $C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ will be denoted by $\{E_{0r}(X), \delta_{0r}\}$ and $\{E_r, \partial_r\}$ respectively.

The grading of these spectral sequences is taken as in (13), although this time we have four gradings: Adams and weight filtration degrees as well as homological and total degrees. From (14) we have that δ_0, and therefore all differentials δ_{0r}, do not change weight filtration nor total degrees; consequently we omit these two gradings in $E_{0r}(X)$ and δ_{0r} takes the form

$$\delta_{0r} : E_{0r}^{is}(X) \to E_{0r}^{i+r,s+1}(X), \tag{16}$$

where i and s stand for Adams filtration and homological degrees respectively. Similar considerations hold for $\{E_r, \partial_r\}$. We study the spectral sequence $\{E_{0r}(X)\}$ at the end of this section, after a careful analysis of the spectral sequence $\{E_r\}$.

As in (15) above, given a filtered object A, we let E_0A denote the associated quotient object. For instance, using the filtrations in $\mathbb{Z}_{(p)}$ and Γ defined in 3.8 we form the associated graded ring $E_0\mathbb{Z}_{(p)}$ and the associated bigraded coalgebra $E_0\Gamma$ over $E_0\mathbb{Z}_{(p)}$. Note that $E_0\mathbb{Z}_{(p)}$ can be identified with the polynomial ring $\mathbb{F}_p[x]$, where x is of filtration degree 1 and corresponds to multiplication by p in $\mathbb{Z}_{(p)}$. On the other hand, $E_0\Gamma$ is bigraded by

$$E_0\Gamma^{ik} = \frac{F^i\Gamma^k}{F^{i+1}\Gamma^k}, \tag{17}$$

which is zero unless $k = jq \ \ (j \geq 0)$ and $i \geq -\nu(j!)$, in which case it is isomorphic to $\mathbb{F}_p$ with generator represented by $p^{i+\nu(j!)}t_j \in \Gamma^k$. Note that each $E_0\Gamma^{*k}$ is a graded module over the (graded) ring $\mathbb{F}_p[x]$, and this is the structure taken in the graded tensor product $E_0\Gamma \otimes_{\mathbb{F}_p[x]} E_0\Gamma$:

$$\big(E_0\Gamma \otimes_{\mathbb{F}_p[x]} E_0\Gamma\big)^{*\,k} = \bigoplus_{k'+k''=k} E_0\Gamma^{*k'} \otimes_{\mathbb{F}_p[x]} E_0\Gamma^{*k''}, \tag{18}$$

where the Adams filtration degree i in a given $E_0\Gamma^{*k'} \otimes_{\mathbb{F}_p[x]} E_0\Gamma^{*k''}$ is the image of the obvious morphism

$$\bigoplus_{i'+i''=i} E_0\Gamma^{i'k'} \otimes_{\mathbb{F}_p} E_0\Gamma^{i''k''} \longrightarrow E_0\Gamma^{*k'} \otimes_{\mathbb{F}_p[x]} E_0\Gamma^{*k''}. \tag{19}$$

Observe also that the bigraded object associated to the tensor product filtration on $\Gamma \otimes \Gamma$,

$$F^i(\Gamma \otimes \Gamma)^k = \bigoplus_{k'+k''=k} \left(\sum_{i'+i''=i} F^{i'}\Gamma^{k'} \otimes F^{i''}\Gamma^{k''} \right), \tag{20}$$

agrees with $E_0\Gamma \otimes_{\mathbb{F}_p[x]} E_0\Gamma$ as described in (18) and (19). Moreover the filtration preserving map $\Delta : \Gamma \to \Gamma \otimes \Gamma$ induces a corresponding map $E_0\Delta : E_0\Gamma \to E_0(\Gamma \otimes \Gamma) = E_0\Gamma \otimes_{\mathbb{F}_p[x]} E_0\Gamma$ which takes the form

$$E_0\Delta(t_j) = \sum t_m \otimes t_{j-m}, \tag{21}$$

where the sum runs over all indices m such that the Adams filtration of $t_m \otimes t_{j-m}$ *equals* that of t_j, or equivalently $\nu\binom{j}{m} = 0$. This determines the coalgebra structure of $E_0\Gamma$. Now by definition $E_0 = E_0C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$, and the above remarks generalize to give an isomorphism of graded $\mathbb{F}_p[x]$–complexes $E_0 \simeq C_{E_0\Gamma}(\mathbb{F}_p[x], \mathbb{F}_p[x])$ (see also the proof of [28, A1.3.9]). In particular we get

$$E_1 \simeq H^*(E_0\Gamma). \tag{22}$$

Definition 4.6. For $i \geq 0$ let Γ_i be the bigraded (by total degree and Adams filtration degree) $\mathbb{F}_p[x]$–coalgebra defined through total degree by $\Gamma_i^k = 0$ except for

$k = jp^iq$ $(0 \le j \le p-1)$, in which case Γ_i^k is $\mathbb{F}_p[x]$–free on a generator τ_{jp^i} of Adams filtration degree $-j(1+p+\cdots+p^{i-1})$. The coproduct is given by

$$\Delta_i(\tau_{jp^i}) = \sum_{m=0}^{j} \tau_{mp^i} \otimes \tau_{(j-m)p^i}. \tag{23}$$

Since the lowest total degree element (other than τ_0) in each Γ_i appears in a large total degree (as i gets large), we can form the tensor product $\bigotimes_{i\ge 0}\Gamma_i$ (over the graded ring $\mathbb{F}_p[x]$).

Proposition 4.7. *As a bigraded $\mathbb{F}_p[x]$–coalgebra, $E_0\Gamma$ is isomorphic to $\bigotimes_{i\ge 0}\Gamma_i$.*

The next lemma is well known, and a proof can be found in [29].

Lemma 4.8. *Let $0 \le m \le j$ be integers with corresponding p–adic decompositions $j = \sum_{k\ge 0} j_k p^k$ and $m = \sum_{k\ge 0} m_k p^k$. Then*

a) $\nu(m!) = \frac{1}{p-1}(m - \sum_{k\ge 0} m_k)$,

b) $\nu\binom{j}{m} = 0$ *if and only if* $m_k \le j_k \ \ \forall k \ge 0$,

c) $\nu\binom{j}{m} = \nu(j) - \nu(m)$ *provided* $m_{\nu(j)} < j_{\nu(j)}$ *and* $m_k = 0$ *for* $k > \nu(j)$.

Proof of 4.7. Defining $\rho(t_a) = \tau_{a_0p^0} \otimes \cdots \otimes \tau_{a_lp^l}$, where $a = a_0p^0 + \cdots + a_lp^l$ is the p–adic decomposition of m, we get a bigraded $\mathbb{F}_p[x]$–isomorphism $E_0\Gamma \xrightarrow{\rho} \bigotimes_{i\ge 0}\Gamma_i$, which does not change Adams filtration degree in view of 4.8. We check that ρ preserves diagonals. For integers $0 \le m \le j$ with p-adic decompositions as in 4.8, the lemma shows that the summation in (21) runs over those m such that $m_k \le j_k \ \ \forall k \ge 0$. On the other hand, by (23) the diagonal in $\bigotimes_{i\ge 0}\Gamma_i$ takes the form

$$\begin{aligned}\Delta(\tau_{j_0p^0} \otimes \cdots \otimes \tau_{j_lp^l}) \\ = \sum_{\substack{0\le k\le l \\ m_k \le j_k}} (\tau_{m_0p^0} \otimes \cdots \otimes \tau_{m_lp^l}) \otimes (\tau_{(j_0-m_0)p^0} \otimes \cdots \otimes \tau_{(j_l-m_l)p^l}),\end{aligned}$$

and the result follows. ∎

Definition 4.9. For $i \ge 0$ let $a_i, b_{i+1} \in C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ be the elements given by $a_i = [t_{p^i}]$ and $b_{i+1} = \partial a_{i+1}$.

Remark 4.10. By (12), b_{i+1} is a sum of elements of the form $-[t_k|t_{p^{i+1}-k}]$ $(0 < k < p^{i+1})$ each of which has Adams filtration $-\nu(k!) - \nu((p^{i+1}-k)!) = \nu\binom{p^{i+1}}{k} - \nu(p^{i+1}!)$. By 4.8 the lowest value for this expression is $1 - \nu(p^{i+1}!)$, which holds only for $k = jp^i$, $1 \le j \le p-1$. In particular the Adams filtration of b_{i+1} is $1 - \nu(p^{i+1}!)$, one higher than that of a_{i+1}.

In view of 4.7 we can rewrite (22) as $E_1 \simeq \bigotimes_{i\ge 0} H^*(\Gamma_i)$. The cohomology of each Γ_i is known to be a tensor product of an exterior algebra with generator represented in $C_{\Gamma_i}(\mathbb{F}_p[x], \mathbb{F}_p[x])$ by $[\tau_{p^i}]$ and a polynomial algebra with generator represented by $[\tau_{p^i}|\tau_{(p-1)p^i}] + \cdots + [\tau_{(p-1)p^i}|\tau_{p^i}]$ (see for instance the proof of [7, Theorem 4.1], where the dual situation is described). Thus 4.10 gives

$$E_1 \simeq \bigotimes_{i\ge 0} \left(E(\alpha_i) \otimes P(\beta_{i+1})\right), \tag{24}$$

where the tensor product is taken over $\mathbb{F}_p[x]$, and α_i and β_{i+1} are represented in $C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ by the elements a_i and b_{i+1} respectively. Since a_0 is a ∂–cycle whereas by definition $\partial a_{i+1} = b_{i+1}$ $(i \geq 0)$, 4.10 implies that a_0 represents a permanent cycle in $\{E_r\}$ and that β_{i+1} is the ∂_1-boundary of α_{i+1} in E_1 (compare with 5.5 and the calculation of differentials in section 5). In particular, the decomposition $E_1 = E(\alpha_0) \otimes \bigotimes_{i\geq 0}(E(\alpha_{i+1}) \otimes P(\beta_{i+1}))$ is one of chain complexes, and since all modules involved are $\mathbb{F}_p[x]$–free we obtain

$$E_2 = E(\alpha_0). \tag{25}$$

Since $E(\alpha_0)$ is all concentrated in homological degree one, the spectral sequence $\{E_r\}$ collapses from its second term. The only nontrivial extensions in E_∞ are given by multiplication by $x \in \mathbb{F}_p[x]$, and we recover the well known cohomology of Γ as the exterior algebra (over $\mathbb{Z}_{(p)}$) on a generator represented by $[t_1] \in C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ (see [7, Theorem 4.1] or [27, Proposition 7.4]).

We use the information above to study the spectral sequence $\{E_{0r}(X)\}$. Since all terms in the spectral sequence $\{E_r\}$ are $\mathbb{F}_p[x]$–free, we have that the spectral sequence associated to the Adams filtration in $\pi_*((\ell \wedge X)^{\langle 0\rangle}) \otimes C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ agrees with $\{E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_r, 1 \otimes \partial_r\}$ (where tensor products are taken over $\mathbb{F}_p[x]$; see the remarks in (17) through (20)). Moreover, since X is $(\ell, H\mathbb{F}_p)$–prime we get

$$E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right) = Ext_{A_p}(H^*((\ell \wedge X)^{\langle 0\rangle}; \mathbb{F}_p); \mathbb{F}_p) \tag{26}$$

(which is an $\mathbb{F}_p$ vector space), where A_p is the mod p Steenrod algebra (we only remark that the gradings in (26) are shifted as usual: in the standard Ext^{st}-grading, s contributes to the Adams filtration degree whereas $t-s$—the homotopy degree—contributes to our total degree). By 4.4 it follows that $E_{00}(X)$ is the subcomplex of $E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_0$ consisting of elements of nonnegative Adams filtration degree; moreover by (16) the differential $1 \otimes \partial_0$, as well as its restriction δ_{00}, do not change Adams filtration degree, and we obtain

Proposition 4.11. *$E_{01}(X)$ is the subcomplex of $E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right) \otimes E_1$ consisting of elements of nonnegative Adams filtration degree.*

Remark 4.12. The next differential δ_{01} increases the Adams filtration degree by one, so that by 4.11 classes in $E_{02}(X)$ arise in two ways: In positive Adams filtration degrees, $E_{01}(X)$ and $E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_1$ have the same homology, while in zero Adams filtration degree all cycles in $E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_1$ (whether or not they are $1 \otimes \partial_1$ boundaries) produce homology classes in $E_{01}(X)$. By (25) the first type of elements are the positive Adams filtration multiples of 1 and α_0 (scalars taken in $E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle}))$ and $\mathbb{F}_p[x]$), whereas the second type of elements are 1 and α_0 themselves with zero Adams filtration coefficients (if any), as well as all $1 \otimes \partial_1$ boundaries of elements in $E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_1$ having Adams filtration -1.

Definition 4.13. A sequence $I = (e_0, e_1, m_1, e_2, m_2, \dots)$ of nonnegative integers, almost all zero and such that $e_i \in \{0, 1\}\ \forall\, i \geq 0$, is called admissible if the following conditions hold:

a) $I \neq (0, 0, 0, \dots), (1, 0, 0, 0, \dots)$,
b) $m_j > 0 \Rightarrow e_k = 1$ for some $k = 1, \dots, j$.

Under such conditions, define the admissible monomials $\varphi^I \in C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ and $\phi^I \in E_1$ by $\varphi^I = a_0^{e_0} a_1^{e_1} b_1^{m_1} a_2^{e_2} b_2^{m_2} \cdots$ and $\phi^I = \alpha_0^{e_0} \alpha_1^{e_1} \beta_1^{m_1} \alpha_2^{e_2} \beta_2^{m_2} \cdots$.

Proposition 4.14. *An $\mathbb{F}_p[x]$–basis for the quotient $E_1/(\partial_1$-boundaries) is given by all admissible monomials ϕ^I together with 1 and α_0; these last two elements are ∂_1-cycles with ∂_1 injective on the submodule generated by the admissible monomials ϕ^I.*

Proof. Since $\partial_1\alpha_{i+1} = \beta_{i+1}$ $i \geq 0$, the ∂_1–boundary of a nonadmissible monomial is a sum of nonadmissible monomials, whereas the ∂_1–boundary of an admissible monomial is the sum of a nonadmissible monomial plus perhaps some admissible ones. Moreover every nonadmissible monomial other than 1 and α_0 is considered in this way once and only once. The result follows. □

Thus, besides the multiples of 1 and α_0, the second type of elements in $E_{02}(X)$ mentioned at the end of 4.12 arise as cycles in $E_{01}(X)$ of the form

$$\gamma\partial_1\phi^I \in E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right) \otimes E_1, \tag{27}$$

where $\gamma \in E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle}))$, $\gamma\phi^I$ is of Adams filtration degree -1 and ϕ^I is an admissible monomial (we omit the tensor product symbol in elements like $\gamma \otimes \partial_1\phi^I$ and $\gamma \otimes \phi^I$). Since multiplication by $x \in \mathbb{F}_p[x]$ increases Adams filtration degree by 1, we get that $x\gamma\partial_1\phi^I = \delta_{01}(x\gamma\phi^I)$ is a boundary in $E_{01}(X)$. Hence multiplication by x is trivial over the elements described in (27). Note also that these elements lie in homological degrees at least 2. The following description of $E_{02}(X)$ results.

Proposition 4.15. *In homological degrees $s \geq 2$ multiplication by x is trivial, and an $\mathbb{F}_p$–basis for $E_{02}(X)$ is given by the elements described in* (27) *with γ running over an $\mathbb{F}_p$–basis of $E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right)$. On the other hand, in homological degrees $s \leq 1$, $E_{02}(X)$ consists of $E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right) \otimes E(\alpha_0)$ (tensor product over $\mathbb{F}_p[x]$).*

The next differentials δ_{0r} $(r \geq 2)$ increase Adams filtration degree at least by 2, and since all elements in $s \leq 1$ are permanent cycles and those in $s \geq 2$ have zero Adams filtration degree, we get that $\{E_{0r}(X)\}$ collapses from its second term, so that 4.15 also gives a description of $E_{0\infty}(X)$. Moreover, by the remarks below (25), the spectral sequence $\{E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle})) \otimes E_r\}$ converges to the cohomology of $\pi_*((\ell \wedge X)^{\langle 0\rangle}) \otimes C_\Gamma(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$, and the above analysis implies that $\{E_{0r}(X)\}$ converges to $E_1(X)$. Representatives in $\mathcal{C}(X) \subseteq C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ for the described homology classes in $E_1(X)$ are as follows: For a given $c \in \pi_*((\ell \wedge X)^{\langle 0\rangle})$ let $\gamma \in E_0(\pi_*((\ell \wedge X)^{\langle 0\rangle}))$ be the associated element; then $c\cdot 1$ and $c\cdot a_0$ represent $\gamma\cdot 1$ and $\gamma \cdot \alpha_0$ respectively and (for suitable γ as in (27)) $c\partial\varphi^I$ represents $\gamma\partial_1\phi^I$ (here $c\partial\varphi^I$ is thought as an element of $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ via the graded isomorphisms in (15)). Since multiplication by $p \in \mathbb{Z}_{(p)}$ corresponds to multiplication by $x \in \mathbb{F}_p[x]$, the considerations after (27) take care of the extensions in $E_{0\infty}(X)$ and we end up with the following description of $E_1(X)$ analogous to that in [19] at the prime two.

Theorem 4.16. *The first term in the WSS for X, $E_1(X)$, is given in homological degrees $s \leq 1$ as $\pi_*((\ell \wedge X)^{\langle 0\rangle}) \otimes E(\alpha_0)$. In homological degrees $s \geq 2$, $E_1(X)$ is $\mathbb{F}_p$–free on generators represented by elements*

$$c\partial\varphi^I \in \mathcal{C}(X) \subseteq C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)}),$$

where $c \in \pi_*((\ell \wedge X)^{\langle 0\rangle})$ runs over a set of representatives for an $\mathbb{F}_p$ basis of $E_0\left(\pi_*((\ell \wedge X)^{\langle 0\rangle})\right)$, $\varphi^I \in C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$ is admissible and $c\varphi^I$ is of Adams filtration -1.

Remark 4.17. In view of 4.10, the Adams filtration degree of $\partial\varphi^I$ is higher by one than that for φ^I, so that (26) and 4.16 suggest writing $E_1(X)$ in homological degrees $s \geq 2$ as

$$\bigoplus_{I\ admissible} Ext_{A_p}^{-AF(\varphi^I)-1}(H^*((\ell \wedge X)^{\langle 0\rangle}; \mathbb{F}_p); \mathbb{F}_p) \cdot \partial\varphi^I,$$

where the degree shown in the Ext functor corresponds to Adams filtration and the factor $\cdot\,\partial\varphi^I$ is used to suggest the representative taken in $\mathcal{C}(X)$.

5. The weight spectral sequence for $R^{\langle i\rangle}$

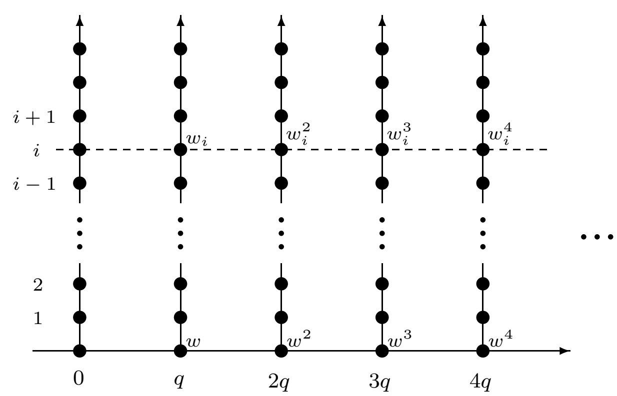
Let B be the suspension spectrum for the p–localization of the classifying space for the symmetric group on p letters, and consider the cofibration

$$B \longrightarrow S^0 \longrightarrow R, \tag{28}$$

where the first arrow represents the Kahn–Priddy map (see [4]). In [22] it is shown that for $p = 2$, $b_0 \wedge R$ splits as a wedge of suspensions of $\mathbb{Z}_{(2)}$ Eilenberg-Mac Lane spectra. The methods can easily be carried over at odd primes to produce a splitting

$$\ell \wedge R \simeq \bigvee_{n\geq 0} \Sigma^{nq} H\mathbb{Z}_{(p)}, \tag{29}$$

where $H\mathbb{Z}_{(p)}$ stands for the $\mathbb{Z}_{(p)}$ Eilenberg-Mac Lane spectrum. In particular, $\ell \wedge R$ is a ring spectrum whose homotopy $\ell_*(R)$ is given as a polynomial ring over $\mathbb{Z}_{(p)}$ on a generator $w \in \ell_q(R)$ of zero Adams filtration. Using the classical Adams spectral sequence together with a standard change–of–rings isomorphism (see for instance [25]), one readily obtains the following chart for $\ell_*(R)$:



as well as the fact that the map $S^0 \to R$ in (28) induces in ℓ_* homology a zero Adams filtration inclusion sending v^n into $p^n w^n$. In particular, this inclusion together with the action of ϕ on $\ell_*(S^0)$ (see (3)) gives the corresponding action of ϕ on $\ell_*(R)$: $\phi(w^r) = \frac{1}{p}(k^r - 1)w^{r-1} = c(\!(\begin{smallmatrix} r \\ 1 \end{smallmatrix})\!)w^{r-1}$, where $c = \frac{k-1}{p}$, which is a unit in $\mathbb{Z}_{(p)}$. Iterating, we obtain

$$\phi^n(w^r) = c^n(\!(\begin{smallmatrix} r \\ 1 \end{smallmatrix})\!) \cdots (\!(\begin{smallmatrix} r-n+1 \\ 1 \end{smallmatrix})\!)w^{r-n} = c^n n!!(\!(\begin{smallmatrix} r \\ n \end{smallmatrix})\!)w^{r-n}. \tag{30}$$

From 3.2 and 3.4 the corresponding chart for $\pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle})$ can be pictured by deleting elements of Adams filtration less that i in the chart above. As suggested there, w_i^m will denote the (zero Adams filtration) generator of $\pi_{mq}((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle})$—thus $w_i^m = p^i w^m$ in $\ell_{mq}(R)$. The isomorphisms in (15) imply that the cobar complex $C_\Gamma(\pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}$–free on generators

$$w_i^m[t_{\overline{n}}], \tag{31}$$

where $m \geq 0$ and $\overline{n} \in \bigcup_{s\geq 0} R_s$ as in 2.2. Defining the (zero Adams filtration) elements $\vartheta_n \in \Gamma$ and $[\vartheta_{\overline{n}}] \in C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$ by $\vartheta_n = n!!c^n t_n$ and $[\vartheta_{\overline{n}}] = [\vartheta_{n_1}|\cdots|\vartheta_{n_s}]$ if $\overline{n} = (n_1, \dots, n_s)$, we see from 3.10 that $\mathcal{C}(R^{\langle i\rangle})$ is $\mathbb{Z}_{(p)}$–free on generators $w_i^m[\vartheta_{\overline{n}}]$; moreover from the easy to check formula $n!!(m-n)!!(\!(\begin{smallmatrix} m \\ n \end{smallmatrix})\!) = m!!$ together with (12) and (30), we get the following expression for the differential in $\mathcal{C}(R^{\langle i\rangle})$:

$$\begin{aligned} d(w_i^m[\vartheta_{\overline{n}}]) = &\sum_{n_0=1}^{m} (\!(\begin{smallmatrix} m \\ n_0 \end{smallmatrix})\!) w_i^{m-n_0}[\vartheta_{n_0}|\vartheta_{\overline{n}}] \\ &+ \sum_{\substack{1\leq e\leq s \\ 0<j<n_e}} (-1)^e (\!(\begin{smallmatrix} n_e \\ j \end{smallmatrix})\!) w_i^m[\cdots|\vartheta_{n_{e-1}}|\vartheta_j|\vartheta_{n_e-j}|\vartheta_{n_{e+1}}|\cdots]. \end{aligned} \tag{32}$$

Definition 5.1. Let Σ denote the filtered graded coalgebra free over $\mathbb{Z}_{(p)}$ on generators $\theta_m \in \Sigma^{mq}$ of zero filtration and diagonal given by the formula $\Delta\theta_m = \sum\limits_{0\leq j\leq m} (\!(\begin{smallmatrix} m \\ j \end{smallmatrix})\!)\theta_j \otimes \theta_{m-j}$.

As in 3.9 we extend the above filtration on Σ to one in $C_\Sigma(\Sigma; \mathbb{Z}_{(p)})$ and refer to it as the Adams filtration. Also, as in 4.1 and its remarks, we have a weight filtration in this complex. The next result follows easily from the considerations above.

Lemma 5.2. *The correspondence $w_i^m[\vartheta_{\overline{n}}] \leftrightarrow \theta_m[\theta_{\overline{n}}]$ defines a chain isomorphism $\mathcal{C}(R^{\langle i\rangle}) \simeq C_\Sigma(\Sigma; \mathbb{Z}_{(p)})$ preserving both Adams and weight filtrations.*

In particular, the obvious inclusion $\mathbb{Z}_{(p)} \hookrightarrow C_\Sigma^{00}(\Sigma; \mathbb{Z}_{(p)})$ gives rise to an augmented complex which as usual is contractible (see for instance the proof of [28, A1.2.12]). Thus although the cohomology of $\mathcal{C}(R^{\langle i\rangle})$ reduces to a single copy of $\mathbb{Z}_{(p)}$, a complete description of the WSS for $R^{\langle i\rangle}$ will be crucial in our analysis of the WSS for $(S^0)^{\langle i\rangle}$ (in the next section). Another consequence of 5.2 is that the WSS for $R^{\langle i\rangle}$ is independent of i, so that in this section we will only consider the case $i = 0$.

Definition 5.3. To keep with the notations in 4.9 and 4.13 we define the following zero Adams filtration elements in $C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$: for $j \geq 0$ we let $u_j = [\vartheta_{p^j}] = p^j!!c^{p^j}a_j$, $v_{j+1} = \frac{1}{p}(p^{j+1})!!c^{p^{j+1}}b_{j+1}$ and for an admissible sequence $I = (e_0, e_1, m_1, e_2, m_2, \dots)$ we let $h^I = u_0^{e_0}u_1^{e_1}v_1^{m_1}u_2^{e_2}v_2^{m_2}\dots = c(I)\varphi^I$, where c is as in (30) and

$$c(I) = \prod_{j\geq 0} \frac{\left(p^j!!\,c^{p^j}\right)^{e_j+m_j}}{p^{m_j}} \quad \text{with} \quad m_0 = 0.$$

Remark 5.4. In terms of this notation, 4.16 gives the following description for $E_1(R)$: In homological degrees $s \leq 1$, $E_1(R) = \ell_*(R) \otimes E(u_0)$, whereas for $s \geq 2$, $E_1(R)$ is $\mathbb{F}_p$–free on generators represented in $\mathcal{C}(R)$ by the elements $\frac{1}{p}w^m\partial h^I$ with I admissible and $m \geq 0$ (since $\partial u_{j+1} = pv_{j+1}$ we see that ∂h^I has Adams filtration 1 and is divisible by p in $C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$, so that in the notation above the factor $\frac{1}{p}$ affects ∂h^I rather than w^m).

Remark 5.5. WSS-differentials δ_r are computed by finding *optimal* representatives for the classes in $E_1(R)$, that is, representatives whose differential d in $\mathcal{C}(R)$ increases the weight filtration as much as possible. Although the differential of the representatives described in 5.4 increases weight filtration by 1, we can usually do better; for instance when $p=3$ the element $z = w^3\partial u_2 - (\!(\begin{smallmatrix}3\\1\end{smallmatrix})\!)w^2[\vartheta_1]u_2 - (\!(\begin{smallmatrix}3\\2\end{smallmatrix})\!)w[\vartheta_2]u_2$ is divisible by 3 in $\mathcal{C}(R)$ (recall $\partial u_2 = 3v_2$) and $\frac{1}{3}z$ represents the same element in $E_1(R)$ as $\frac{1}{3}w^3\partial u_2$ does (they have weight filtration 9); moreover the formula $d(w^3u_2) = (\!(\begin{smallmatrix}3\\1\end{smallmatrix})\!)w^2[\vartheta_1]u_2 + (\!(\begin{smallmatrix}3\\2\end{smallmatrix})\!)w[\vartheta_2]u_2 + [\vartheta_3]u_2 - w^3\partial u_2$ implies $d(z) = d([\vartheta_3]u_2) = \partial(u_1u_2)$. Since these elements are divisible by 3 in $\mathcal{C}(R)$ and this complex has no torsion, we obtain $d(\frac{1}{3}z) = \frac{1}{3}\partial(u_1u_2)$, which has weight filtration 12. This means that $\frac{1}{3}w^3\partial u_2$ survives to an element in $E_3(R)$ potentially yielding the δ_3–differential $\frac{1}{3}w^3\partial u_2 \mapsto \frac{1}{3}\partial(u_1u_2)$. Such a differential might fail to hold if either one of the elements involved dies in a previous stage of the WSS, in which case we should proceed to find either a lower WSS-differential hitting $\frac{1}{3}w^3\partial u_2$ or a (necessarily better) representative for this element having a d-differential with a higher weight filtration.

Notation 5.6. Throughout the rest of the paper unless otherwise specified, I will stand for an admissible sequence and $k(I)$ will denote the first nonnegative integer j such that $e_j = 1$, where $I = (e_0, e_1, m_1, e_2, m_2, \dots)$. If no confusion arises, $k(I)$ will simply be denoted by k.

Definition 5.7. Define the elements $D_jw^mh^I \in \mathcal{C}(R)$ ($j = 0,1$), by the formulas $d(w^mh^I) = D_0w^mh^I + D_1w^mh^I$ and

$$D_1w^mh^I \quad = \sum_{\nu\binom{m}{j}=0} (\!(\begin{smallmatrix}m\\j\end{smallmatrix})\!)w^{m-j}[\vartheta_j]h^I.$$

The next result will produce as in 5.5 a family of differentials in the WSS for R. Recall from 4.1 that the weight filtration of an element x is denoted by $WF(x)$.

Lemma 5.8. *For $\nu(m) < k(I)$, the following equation holds in $\mathcal{C}(R)$ modulo elements of weight filtration larger than $WF(u_{\nu(m)}h^I)$:*

$$dD_0w^mh^I = -(\!(\begin{smallmatrix}m\\p^{\nu(m)}\end{smallmatrix})\!)w^{m-p^{\nu(m)}}\partial\left(u_{\nu(m)}h^I\right).$$

Proof. By 4.8 the lowest weight filtration summand in the expression for $D_1w^mh^I$ is $(\!(\begin{smallmatrix}m\\p^{\nu(m)}\end{smallmatrix})\!)w^{m-p^{\nu(m)}}[\vartheta_{p^{\nu(m)}}]h^I = (\!(\begin{smallmatrix}m\\p^{\nu(m)}\end{smallmatrix})\!)w^{m-p^{\nu(m)}}u_{\nu(m)}h^I$; so, modulo weight filtration larger than $WF(u_{\nu(m)}h^I)$, we have

$$\begin{aligned} dD_0w^mh^I &= -dD_1w^mh^I = -d\left((\!(\begin{smallmatrix}m\\p^{\nu(m)}\end{smallmatrix})\!)w^{m-p^{\nu(m)}}u_{\nu(m)}h^I\right) \\ &= -(\!(\begin{smallmatrix}m\\p^{\nu(m)}\end{smallmatrix})\!)w^{m-p^{\nu(m)}}\partial(u_{\nu(m)}h^I). \end{aligned}$$

□

Remark 5.9. In view of 5.4 both $D_0w^mh^I$ and $w^m\partial h^I$ have Adams filtration 1 and are divisible by p in $\mathcal{C}(R)$ (uniquely divisible since $\mathcal{C}(R)$ is $\mathbb{Z}_{(p)}$–free). Moreover, they agree modulo elements of weight filtration larger than $WF(\partial h^I)$ in view of (12), and so their $\frac{1}{p}$–multiples represent the same element in the WSS for R, which by 5.8 survives to the term $E_{p^{\nu(m)}}(R)$, potentially producing (up to units) the $\delta_{p^{\nu(m)}}$–differential $\frac{1}{p}w^m\partial h^I \mapsto \frac{1}{p}w^{m-p^{\nu(m)}}\partial(u_{\nu(m)}h^I)$ (in view of 5.8, the target of such a differential represents a permanent cycle in the WSS). As in 5.5 this differential could still fail in case, for instance, the permanent cycle is killed by an earlier differential (we will see this is not the case). Note finally that the hypothesis $\nu(m) < k(I)$ is not essential in the proof of 5.8; it is only used to make sure $u_{\nu(m)}h^I$ is admissible, so that the above differential does land in one of the elements in $E_1(R)$ (see 5.4). We will see later in this section that for $\nu(m) \geq k(I)$, the element $\frac{1}{p}w^m\partial h^I$ survives to a higher term in the spectral sequence, where it supports (either as target or as source) another type of differential. A concrete example of this is given below; it is included in part to motivate Definitions 5.11, 5.12 and 5.14.

Example 5.10. Let $p = 3$. The remarks above suggest the "differential"

$$\delta_1(\frac{1}{3}w^2\partial(u_0u_1)) = \frac{1}{3}w\partial(u_0u_0u_1)$$

in the WSS for R; however, $u_0u_0u_1$ is not an admissible monomial, in fact (24) and the methods in the previous section imply that $\frac{1}{3}w\partial(u_0u_0u_1)$ produces a trivial class in $E_1(R)$, so that $\frac{1}{3}w^2\partial(u_0u_1)$ survives to $E_2(R)$, and we next indicate how it produces a δ_2–differential. Let $z = w^2\partial(u_0u_1) - w[\vartheta_2]\partial u_1$. Since the second summand of z has weight filtration 5, we see that $\frac{1}{3}z$ represents in the WSS for R the same element as $\frac{1}{3}w^2\partial(u_0u_1)$ does (recall $\partial u_1 = 3v_1$ and $\partial u_0 = 0$). Moreover, from (12) and (32)

$$\begin{aligned}
d(z) &= d(w^2\partial(u_0u_1)) - d(w[\vartheta_2]\partial u_1)\\
&= (\!(\tbinom{2}{1})\!)w[\vartheta_1]\partial(u_0u_1) + [\vartheta_2]\partial(u_0u_1) - [\vartheta_1|\vartheta_2]\partial u_1 + (\!(\tbinom{2}{1})\!)w[\vartheta_1|\vartheta_1]\partial u_1\\
&= -(\!(\tbinom{2}{1})\!)w[\vartheta_1]u_0\partial u_1 - [\vartheta_2]u_0\partial u_1 - [\vartheta_1|\vartheta_2]\partial u_1 + (\!(\tbinom{2}{1})\!)w[\vartheta_1|\vartheta_1]\partial u_1\\
&= (-[\vartheta_2|\vartheta_1] - [\vartheta_1|\vartheta_2])\partial u_1 \qquad (\text{recall that} \quad [\vartheta_1] = u_0)\\
&= 3(\!(\tbinom{3}{1})\!)^{-1}(\partial[\vartheta_3])v_1 \qquad \big(\text{note that} \quad (\!(\tbinom{3}{1})\!) = (\!(\tbinom{3}{2})\!)\big)\\
&= \mathit{unit}\cdot(\partial u_1)v_1\\
&= \mathit{unit}\cdot\partial(u_1v_1),
\end{aligned}$$

so that $d(\frac{1}{3}z) = \mathit{unit}\cdot\frac{1}{3}\partial(u_1v_1)$. Thus $\frac{1}{3}\partial(u_1v_1)$ represents a permanent cycle in the WSS (u_1v_1 is admissible), and since $WF(u_0u_1) = 4$ and $WF(u_1v_1) = 6$, we get the δ_2–differential $\frac{1}{3}w^2\partial(u_0u_1) \mapsto \frac{1}{3}\partial(u_1v_1)$.

Definition 5.11. For a positive integer m let $c(m)$ be defined by the relations $0 < c(m) < p$ and $m \equiv c(m)p^{\nu(m)} \pmod{p^{\nu(m)+1}}$.

Definition 5.12. For positive integers m and λ with $\nu(m) = k$, $c(m) = p-1$ and $0 < \lambda < p$ let

a) $a_{\lambda m} = \left(\!\!\left(\tbinom{m}{(\lambda-1)p^k}\right)\!\!\right)\left(\!\!\left(\tbinom{\lambda p^k}{p^k}\right)\!\!\right)^{-1}\left(\!\!\left(\tbinom{m-(\lambda-1)p^k}{(p-\lambda)p^k}\right)\!\!\right)$,

b) $a_m = pa_{\lambda m}\left(\!\!\left(\tbinom{p^{k+1}}{\lambda p^k}\right)\!\!\right)^{-1}$.

The elements $a_{\lambda m}$ and a_m are units in $\mathbb{Z}_{(p)}$ in view of 4.8. The following result shows a_m is well defined. The proof is straightforward, and so is omitted.

Lemma 5.13. *For $\nu(m) = k$, $c(m) = p-1$ and $0 < \lambda \leq r < p$*

a) $a_{\lambda m}\left(\!\!\left(\tbinom{p^{k+1}}{\lambda p^k}\right)\!\!\right)^{-1} = a_{rm}\left(\!\!\left(\tbinom{p^{k+1}}{rp^k}\right)\!\!\right)^{-1}$,

b) $\left(\!\!\left(\tbinom{m}{(\lambda-1)p^k}\right)\!\!\right)\left(\!\!\left(\tbinom{\lambda p^k}{p^k}\right)\!\!\right)^{-1}\left(\!\!\left(\tbinom{m-(\lambda-1)p^k}{(r-\lambda)p^k}\right)\!\!\right) = \left(\!\!\left(\tbinom{m}{(r-1)p^k}\right)\!\!\right)\left(\!\!\left(\tbinom{rp^k}{p^k}\right)\!\!\right)^{-1}\left(\!\!\left(\tbinom{rp^k}{\lambda p^k}\right)\!\!\right)$.

Definition 5.14. a) For an admissible sequence I with h^I starting in the form $h^I = u_k u_{k+d} H$ (where H is a monomial in u's and v's), define the derivative I' of I by $h^{I'} = u_{k+1} v_{k+d} H$. Note that I' is again admissible. Define also the element $g^{I'} \in C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$ by the relation (recall $\partial u_{j+1} = p v_{j+1}$)

$$\frac{1}{p}\partial(u_{k+1}u_{k+d}H) = v_{k+1}u_{k+d}H - g^{I'}. \tag{33}$$

b) Define the elements $D_{0j}u_{k+1} \in C_\Gamma(\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)})$ $(j = 0, 1)$ by the formulas $v_{k+1} = D_{00}u_{k+1} + D_{01}u_{k+1}$ and

$$D_{01}u_{k+1} = -\sum_{j=1}^{p-1} \frac{1}{p}\left(\!\binom{p^{k+1}}{jp^k}\!\right)[\vartheta_{(p-j)p^k}|\vartheta_{jp^k}].$$

c) Define the elements $D'_j w^m[\vartheta_a]h^I \in \mathcal{C}(R)$ $\ (j = 0, 1)$ by the formulas

$$d(w^m[\vartheta_a]h^I) = D'_0 w^m[\vartheta_a]h^I + D'_1 w^m[\vartheta_a]h^I,$$

$$D'_1 w^m[\vartheta_a]h^I = \sum_{\nu\binom{m}{j}=0} \left(\!\binom{m}{j}\!\right) w^{m-j}[\vartheta_j|\vartheta_a]h^I - \sum_{\nu\binom{a}{j}=0} \left(\!\binom{a}{j}\!\right) w^m[\vartheta_j|\vartheta_{a-j}]h^I.$$

d) For I as in a) and m such that $\nu(m) = k(I) = k$ and $c(m) = p-1$, define the elements $\overline{D}_j w^m h^I \in \mathcal{C}(R)$ $(j = 0, 1)$ by

$$\begin{aligned}\overline{D}_j w^m h^I \quad &= \quad \frac{1}{a_m}\sum_{\lambda=1}^{p-1}\left(\!\binom{m}{(\lambda-1)p^k}\!\right)\left(\!\binom{\lambda p^k}{p^k}\!\right)^{-1} D'_j w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}\\ &\quad + (-1)^j w^{m-(p-1)p^k}(D_{00}u_{k+1})h^{I-\Delta_k},\end{aligned}$$

where $h^{I-\Delta_k}$ is the admissible monomial given by $h^I = u_k h^{I-\Delta_k}$.

Lemma 5.15. *For m and I as in 5.14 d), the following equation holds in $\mathcal{C}(R)$ modulo elements of weight filtration larger than $WF(g^{I'})$:*

$$d\overline{D}_0 w^m h^I = w^{m-(p-1)p^k}\partial g^{I'}.$$

Proof. Let $\sigma = WF(h^{I-\Delta_k})$, so that $WF(g^{I'}) = p^{k+1}+\sigma$. In what follows, equalities modulo weight filtration larger than $p^{k+1}+\sigma$ are denoted by the symbol "$\equiv$". For $0 < \lambda < p$, 4.8 gives

$$\begin{aligned}&(\!\binom{m}{(\lambda-1)p^k}\!)(\!\binom{\lambda p^k}{p^k}\!)^{-1}D'_1 w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}\\ &\equiv \sum_{\mu=1}^{p-\lambda}(\!\binom{m}{(\lambda-1)p^k}\!)(\!\binom{\lambda p^k}{p^k}\!)^{-1}(\!\binom{m-(\lambda-1)p^k}{\mu p^k}\!)w^{m-(\lambda+\mu-1)p^k}[\vartheta_{\mu p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k}\\ &\quad -\sum_{\mu=1}^{\lambda-1}(\!\binom{m}{(\lambda-1)p^k}\!)(\!\binom{\lambda p^k}{p^k}\!)^{-1}(\!\binom{\lambda p^k}{\mu p^k}\!)w^{m-(\lambda-1)p^k}[\vartheta_{(\lambda-\mu)p^k}|\vartheta_{\mu p^k}]h^{I-\Delta_k}.\end{aligned}$$

Therefore the component of $\sum_{\lambda=1}^{p-1}(\!\binom{m}{(\lambda-1)p^k}\!)(\!\binom{\lambda p^k}{p^k}\!)^{-1}D'_1 w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}$ in weight filtration $rp^k+\sigma$ $(2 \le r \le p)$ is

$$\sum_{\lambda=1}^{r-1}(\!\binom{m}{(\lambda-1)p^k}\!)(\!\binom{\lambda p^k}{p^k}\!)^{-1}(\!\binom{m-(\lambda-1)p^k}{(r-\lambda)p^k}\!)w^{m-(r-1)p^k}[\vartheta_{(r-\lambda)p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k},$$

minus, when $r < p$,

$$\sum_{\mu=1}^{r-1} (\!(\,{}^{m}_{(r-1)p^k}\,)\!)(\!(\,{}^{rp^k}_{p^k}\,)\!)^{-1}(\!(\,{}^{rp^k}_{\mu p^k}\,)\!) w^{m-(r-1)p^k}[\vartheta_{(r-\mu)p^k}|\vartheta_{\mu p^k}]h^{I-\Delta_k},$$

and any other components lie in weight filtration larger than $p^{k+1}+\sigma$. Thus we get by 5.13 b)

$$\begin{aligned}\sum_{\lambda=1}^{p-1} (\!(\,{}^{m}_{(\lambda-1)p^k}\,)\!)(\!(\,{}^{\lambda p^k}_{p^k}\,)\!)^{-1} D_1' w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k} \\ \equiv \sum_{\lambda=1}^{p-1} a_{\lambda m} w^{m-(p-1)p^k}[\vartheta_{(p-\lambda)p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k}.\end{aligned} \tag{34}$$

Multiplying by $\frac{1}{a_m}$ and taking differentials, we get

$$\begin{aligned} & d\Bigg(\frac{1}{a_m}\sum_{\lambda=1}^{p-1} (\!(\,{}^{m}_{(\lambda-1)p^k}\,)\!)(\!(\,{}^{\lambda p^k}_{p^k}\,)\!)^{-1} D_0' w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}\Bigg) \\ \equiv\ & -d\Bigg(\frac{1}{a_m}\sum_{\lambda=1}^{p-1} (\!(\,{}^{m}_{(\lambda-1)p^k}\,)\!)(\!(\,{}^{\lambda p^k}_{p^k}\,)\!)^{-1} D_1' w^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}\Bigg) \\ \equiv\ & -d\Bigg(\sum_{\lambda=1}^{p-1} \tfrac{1}{p}\,(\!(\,{}^{p^{k+1}}_{\lambda p^k}\,)\!) w^{m-(p-1)p^k}[\vartheta_{(p-\lambda)p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k}\Bigg) \\ \equiv\ & d\Bigg(w^{m-(p-1)p^k}(D_{01}u_{k+1})h^{I-\Delta_k}\Bigg) \\ \equiv\ & d\Bigg(w^{m-(p-1)p^k}(v_{k+1}-D_{00}u_{k+1})h^{I-\Delta_k}\Bigg),\end{aligned}$$

so that $d(\overline{D}_0 w^m h^I) \equiv d(w^{m-(p-1)p^k}v_{k+1}h^{I-\Delta_k}) \equiv w^{m-(p-1)p^k}\partial\left(v_{k+1}h^{I-\Delta_k}\right)$, which by 5.14 a) implies the result. □

Remark 5.16. By 5.4 both $\overline{D}_0 w^m h^I$ and $w^m\partial h^I$ have Adams filtration 1 and are divisible by p in $\mathcal{C}(R)$; moreover they agree (up to units) modulo elements of weight filtration larger than $WF(h^I)$ (from its definition, the lowest weight filtration term in $\overline{D}_0 w^m h^I$ is carried by

$$\frac{1}{a_m}D_0' w^m[\vartheta_{p^k}]h^{I-\Delta_k} = \frac{1}{a_m}D_0 w^m[\vartheta_{p^k}]h^{I-\Delta_k} = \frac{1}{a_m}D_0 w^m h^I,$$

which, as in 5.9, agrees up to units with $w^m\partial h^I$ modulo weight filtration larger than $WF(h^I)$); thus their $\frac{1}{p}$–multiple represent (up to units) the same element in the WSS for R, which by 5.15 survives to the term $E_{(p-1)p^k}(R)$, potentially producing up to units the $\delta_{(p-1)p^k}$–differential $\frac{1}{p}w^m\partial h^I \mapsto \frac{1}{p}w^{m-(p-1)p^k}\partial g^{I'}$. In 5.20 below we make the arrangements needed to identify the target $\frac{1}{p}w^{m-(p-1)p^k}\partial g^{I'}$ of this differential in terms of the basis for $E_1(R)$ given in 5.4.

Recall from the remarks after (31) that $\mathcal{C}(R)$ is $\mathbb{Z}_{(p)}$–free, so that inverting p produces a monomorphism $\mathcal{C}(R) \hookrightarrow \mathcal{C}(R)\otimes\mathbb{Q}$ of chain complexes.

Definition 5.17. Let $m \geq 0$. For $0 \leq j < \nu(m+1)$ define $c_j \in \mathbb{Q}$ and elements A_j, $B_j \in \mathcal{C}(R) \otimes \mathbb{Q}$ (depending on m too) by

$$c_j = \frac{1}{p^{\nu(m+1)-j-1}} \left(\!\!\binom{m+1}{p^{j+1}}\!\!\right), \qquad \frac{1}{p^{\nu(m+1)-j}} d(w^{m+1} \cdot 1) = A_j + B_j$$

and

$$B_j = \frac{1}{p^{\nu(m+1)-j}} \sum_{r \geq p^{j+1}} \left(\!\!\binom{m+1}{r}\!\!\right) w^{m+1-r} [\vartheta_r].$$

Note that c_j is a unit in $\mathbb{Z}_{(p)}$ in view of 4.8.

Lemma 5.18. a) *$d(w^{m+1} \cdot 1) = (\!\binom{m+1}{1}\!) w^m u_0$ modulo elements of weight filtration larger than* 1.

b) *For $0 \leq j < \nu(m+1)$, A_j lies in $\mathcal{C}(R)$ and its differential agrees with $-c_j \frac{1}{p} w^{m+1-p^{j+1}} \partial u_{j+1}$ modulo elements of weight filtration larger than p^{j+1}.*

Proof. Everything follows by its definitions and 4.8, except for the formula in b), which is a consequence of the fact that $d(A_j) = -d(B_j)$ and that the lowest weight filtration term in B_j is $\frac{1}{p^{\nu(m+1)-j}} (\!\binom{m+1}{p^{j+1}}\!) w^{m+1-p^{j+1}} u_{j+1}$. □

Remark 5.19. Recall from 5.4 that in homological degrees $s \leq 1$, $E_1(R)$ is $\mathbb{Z}_{(p)}$–free with basis the elements $w^m u_0^{\varepsilon}$ $(m \geq 0, \varepsilon = 0, 1)$. By 5.18 a) we get up to units the WSS-differential for R : $\delta_1(w^{m+1} \cdot 1) = p^{\nu(m+1)} w^m u_0$. Moreover for $0 \leq j < \nu(m+1)$, 5.18 b) implies that $\frac{1}{p} w^{m+1-p^{j+1}} \partial u_{j+1}$ is represented by a d–boundary in $\mathcal{C}(R)$, thus producing a permanent cycle in the spectral sequence. Since A_j and $p^j w^m u_0$ represent up to units the same element in the WSS for R, 5.18 b) also shows that $p^j w^m u_0$ survives to $E_{p^{j+1}-1}(R)$, potentially producing up to units the $\delta_{p^{j+1}-1}$-differential $p^j w^m u_0 \mapsto \frac{1}{p} w^{m+1-p^{j+1}} \partial u_{j+1}$.

Remark 5.20. Before describing in full the WSS for R we need to make some minor adjustments in the way we detect elements in $E_1(R)$ arising in homological degrees $s \geq 2$, that is, of the form $\frac{1}{p} w^m \partial h^I$ (see 5.4). Such elements were analyzed through the auxiliary spectral sequence $\{E_{0r}(R), \delta_{0r}\}$ together with the basis of $E_1/(\partial_1 - \mathit{boundaries})$ consisting of the admissible monomials ϕ^I. Since δ_0 (and therefore all differentials δ_{0r}) are independent of the coefficients $w^m \in \ell_*(R)$ (see (15) and the remarks before it), the methods in the previous section work perfectly well if we use a different basis of $E_1/(\partial_1 - \mathit{boundaries})$ for each such coefficient w^m. This is done in order to identify in a natural way the targets of the differentials suggested by 5.16. In detail, let $m \geq 0$. For an admissible monomial h^J starting as $h^J = u_{k+1} v_{k+d} \dots$ with $\nu(m) \geq k(J)$, consider the element g^J as defined in 5.14 a) (note that such an admissible J is the derivative of a uniquely defined admissible sequence I) and define the corresponding element $\Phi^J \in E_1$ by replacing the u's and v's appearing in the expression for g^J by α's and β's respectively, so that g^J represents the $c(J)$–multiple of Φ^J in E_1 (recall from 5.3 that $h^J = c(J)\varphi^J$, so that in the expression "$c(J)$–multiple" we must take into account that E_1 is an $\mathbb{F}_p[x]$–module with x corresponding to multiplication by p). From its definition $\Phi^J - \phi^J$ is a (possibly empty) sum of monomials each of which by (24) is trivial in E_1 (if $d = 1$) or is an admissible monomial starting with a double "α". Thus replacing each such ϕ^J by the corresponding Φ^J produces a new basis for $E_1/(\partial_1 - \mathit{boundaries})$, and as explained above, this is the basis we use for the coefficient w^m in the analysis of the previous section. The result is a new $\mathbb{F}_p$ basis for $E_1(R)$ in homological degrees $s \geq 2$, which can be described as follows (as usual, I is an admissible sequence):

I. Elements of the form $\frac{1}{p}w^m\partial h^I$ with $\nu(m) < k(I)$.

II. Elements of the form $\frac{1}{p}w^m\partial h^I$ with h^I starting as $h^I = u_k u_{k+d}\cdots$, $\nu(m) \geq k(I)$ and, in case $\nu(m) = k(I)$, $c(m) < p-1$.

III. Elements of the form $\frac{1}{p}w^m\partial h^I$ with h^I starting as $h^I = u_k u_{k+d}\cdots$, $\nu(m) = k(I)$ and $c(m) = p-1$.

IV. Elements of the form $\frac{1}{p}w^m\partial g^I$ with h^I starting as $h^I = u_{k+1}v_{k+d}^e\cdots$, $e > 0$ and $\nu(m) \geq k(I)$.

V. Elements of the form $\frac{1}{p}w^m\partial u_k$ with $\nu(m) \geq k \geq 1$.

Here we have used representatives in $\mathcal{C}(R)$ to denote the corresponding classes in $E_1(R)$ (compare with 4.17). This is also the case in the next result.

Theorem 5.21. *The following are, up to units, all differentials in the WSS for R:*

a) $\delta_{p^{\nu(m)}}\left(\frac{1}{p}w^m\partial h^I\right) = \frac{1}{p}w^{m-p^{\nu(m)}}\partial(u_{\nu(m)}h^I)$ *for* $\frac{1}{p}w^m\partial h^I$ *of type* 5.20 I.

b) $\delta_{(p-1)p^{\nu(m)}}\left(\frac{1}{p}w^m\partial h^I\right) = \frac{1}{p}w^{m-(p-1)p^{\nu(m)}}\partial g^{I'}$ *for* $\frac{1}{p}w^m\partial h^I$ *of type* 5.20 III.

c) $\delta_1(p^e\, w^{m+1}\cdot 1) = p^{e+\nu(m+1)}w^m u_0$ *for* $e, m \geq 0$.

d) $\delta_{p^{j+1}-1}(p^j w^m u_0) = \frac{1}{p}w^{m+1-p^{j+1}}\partial u_{j+1}$ *for* $0 \leq j < \nu(m+1)$.

These differentials wipe out all elements in the WSS, except for the multiples of $w^0\cdot 1$, which produce a copy of $\mathbb{Z}_{(p)}$ in the cohomology of $\mathcal{C}(R)$.

Remark 5.22. In order to analyze differentials in the WSS for R as in 5.5, it will be convenient to make a summary of the representatives suggested by 5.8, 5.15 and 5.18 for the basis described in 5.20:

i) A representative $\frac{1}{p}w^m\partial h^I$ of type 5.20 I is replaced by $\frac{1}{p}D_0w^mh^I$; the corresponding representative $\frac{1}{p}w^{m-p^{\nu(m)}}\partial(u_{\nu(m)}h^I)$ of type 5.20 II is replaced by $\frac{1}{p}dD_0w^mh^I$ (see 5.9).

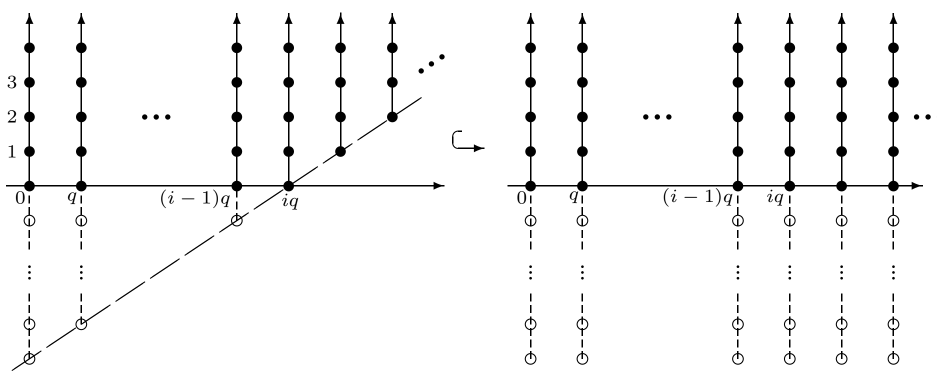
ii) A representative $\frac{1}{p}w^m\partial h^I$ of type 5.20 III is replaced by $\frac{1}{p}\overline{D}_0w^mh^I$; the corresponding representative $\frac{1}{p}w^{m-(p-1)p^{\nu(m)}}\partial g^{I'}$ of type 5.20 IV is replaced by $\frac{1}{p}d\overline{D}_0w^mh^I$ (see 5.16).

iii) Multiples $p^aw^{m+1}\cdot 1 \in E_1(R)$ tautologically represent themselves. Multiples $p^jw^mu_0 \in E_1(R)$ are represented (up to units) by $d(p^{j-\nu(m+1)}w^{m+1}\cdot 1)$ if $j \geq \nu(m+1)$ and by A_j if $0 \leq j < \nu(m+1)$, and in the last case the representative $\frac{1}{p}w^{m+1-p^{j+1}}\partial u_{j+1}$ of type 5.20 V is replaced by $d(A_j)$ (see 5.19).

Proof of 5.21. We have already seen that the use of the representatives in 5.22 suggests such differentials. Note that a) sets a 1-1 correspondence between elements of type 5.20 I and elements of type 5.20 II; likewise, b) sets a 1-1 correspondence between elements of type 5.20 III and elements of type 5.20 IV. Moreover every element on the right of d) is of type 5.20 V, and by rewriting $\frac{1}{p}w^m\partial u_k = \frac{1}{p}w^{m'+1-p^{j+1}}\partial u_{j+1}$ with $j = k-1$ and $m' = m + p^k - 1$ we see that every element of type 5.20 V appears as an element on the right of d). Similarly the (zero Adams filtration $\mathbb{Z}_{(p)}$-multiples of the) elements involved in c) and on the left of d) take into account once and only once every element in $E_1(R)$ of homological degree $s \leq 1$ (except for the $\mathbb{Z}_{(p)}$ multiples of $w^0 \cdot 1$). Therefore there is no room for the anomalous behavior described at the end of 5.5 for any more differentials. □

6. The weight spectral sequence for $S^{\langle i\rangle}$

In this section we obtain a description of the cohomology of $\mathcal{C}(S^{\langle i\rangle})$ (where $S^{\langle i\rangle} = (S^0)^{\langle i\rangle}$) from our detailed understanding of the WSS for $R^{\langle i\rangle}$. As remarked at the beginning of section 5, the map $S^0 \to R$ in (28) induces Adams–filtration preserving inclusions $\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}) \hookrightarrow \pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle})$ which can be pictured in the following charts:



As in 3.2, Adams projections induce inclusions

$$\pi_*((\ell \wedge S^{\langle i+1\rangle})^{\langle 0\rangle}) \hookrightarrow \pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle})$$

and

$$\pi_*((\ell \wedge R^{\langle i+1\rangle})^{\langle 0\rangle}) \hookrightarrow \pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle}),$$

which in turn produce inclusions of chain complexes

$$\begin{array}{ccccccc}
\mathcal{C}(R^{\langle i+1\rangle}) & \hookrightarrow & \mathcal{C}(R^{\langle i\rangle}) & & C_\Gamma(\pi_*((\ell \wedge R^{\langle i+1\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)}) & \hookrightarrow & C_\Gamma(\pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{C}(S^{\langle i+1\rangle}) & \hookrightarrow & \mathcal{C}(S^{\langle i\rangle}) & & C_\Gamma(\pi_*((\ell \wedge S^{\langle i+1\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)}) & \hookrightarrow & C_\Gamma(\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)})
\end{array}$$

and since $\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}) \to \pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle})$ has zero Adams filtration we see from 3.10 that

$$\mathcal{C}(S^{\langle i\rangle}) = \mathcal{C}(R^{\langle i\rangle}) \cap C_\Gamma(\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)}). \tag{35}$$

By the coefficient of a monomial in $C_\Gamma(\pi_*((\ell \wedge X)^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ we mean the $\pi_*((\ell \wedge X)^{\langle 0\rangle})$ portion appearing on the right hand side of the second isomorphism in (15). Thus determining whether or not an element in $C_\Gamma(\pi_*((\ell \wedge R^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ belongs to $C_\Gamma(\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)})$ reduces to a question about coefficients, which, in view of the charts, is settled by

$$p^a w_i^m \in \pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}) \quad \text{if and only if} \quad a + i \geq m \tag{36}$$

and in particular we see from (31) that $\mathcal{C}(S^{\langle i\rangle})$ is $\mathbb{Z}_{(p)}$–free on generators $p^{a_0}w_i^m[\vartheta_{\overline{n}}]$ where $a_0=\max\{0,m-i-\nu(\overline{n}!)\}$. Observe that the equation $\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle})=\pi_*((\ell\wedge R^{\langle i\rangle})^{\langle 0\rangle})\cap\ell_*(S^0)$ implies the following refinement of (35):

$$\mathcal{C}(S^{\langle i\rangle})=\mathcal{C}(R^{\langle i\rangle})\cap C_\Gamma(\ell_*(S^0);\mathbb{Z}_{(p)}). \tag{37}$$

On the other hand, the inclusion $\mathcal{C}(S^{\langle i\rangle})\hookrightarrow\mathcal{C}(R^{\langle i\rangle})$ induces a map of spectral sequences $\{E_r(S^{\langle i\rangle})\}\to\{E_r(R^{\langle i\rangle})\}$, which is injective for $r=1$ in view of 4.16. As with $R^{\langle i\rangle}$, we change the given basis in $E_1(S^{\langle i\rangle})$ to obtain the following description of these groups:

Description 6.1. In homological degrees $s\le 1$, $E_1(S^{\langle i\rangle})$ is $\mathbb{Z}_{(p)}$–free on generators $p^{\max\{0,m-i\}}w_i^m u_0^\varepsilon$ ($\varepsilon=0,1$), whereas for $s\ge 2$ it is $\mathbb{F}_p$–free on generators represented by the elements of types I through V in 5.20 (replacing w^m by w_i^m in view of 5.2) lying in $\mathcal{C}(S^{\langle i\rangle})$, or equivalently whose coefficient lies in $\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle})$. In view of (36) and 5.3, a typical element $\frac{1}{p}w_i^m\partial h^I$ (or $\frac{1}{p}w_i^m\partial g^I$) represents a class in $E_1(S^{\langle i\rangle})$ if and only if

$$\nu(I)+i-1\ge m \tag{38}$$

where $\nu(I)$ stands for $\nu(c(I))$.

Remark 6.2. We just observed the equivalence between $\frac{1}{p}w_i^m\partial h^I\in\mathcal{C}(S^{\langle i\rangle})$ and $\frac{1}{p}c(I)w_i^m\in\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle})$. In fact, since every summand in the expression (12) for $d(\frac{1}{p}w_i^m h^I)$ comes either from the $1\otimes\partial$ portion of d, which is independent of the coefficient $\frac{1}{p}c(I)w_i^m$, or from the action of the powers of ϕ on this coefficient, it follows that each such summand lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$, provided $\frac{1}{p}c(I)w_i^m\in\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle})$. In particular, $\frac{1}{p}D_jw_i^mh^I\in C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$ $(j=0,1)$ as long as $\frac{1}{p}w_i^m\partial h^I\in\mathcal{C}(S^{\langle i\rangle})$. Likewise if $c\,w_i^m[\vartheta_n]h^I$ lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$ for some $c\in\mathbb{Z}_{(p)}$, then so does $cD'_jw_i^m[\vartheta_n]h^I$ $(j=0,1)$.

Lemma 6.3. *Let* $\xi=\frac{1}{p}w_i^m\partial h^I$, $\zeta=\frac{1}{p}w_i^{m-p^{\nu(m)}}\partial(u_{\nu(m)}h^I)$ *and assume* ξ *is an element of type* 5.20 I.

a) *If* ξ *represents a class in* $E_1(S^{\langle i\rangle})$ *then so does* ζ*; moreover* $\frac{1}{p}D_0w_i^mh^I$ *lies in* $\mathcal{C}(S^{\langle i\rangle})$.

b) *If* ζ *represents a class in* $E_1(S^{\langle i\rangle})$ *then both* $\frac{1}{p}dD_0w_i^mh^I$ *and* $D_1w_i^mh^I$ *lie in* $\mathcal{C}(S^{\langle i\rangle})$*, and if in addition* $i\ge 1$ *then* $\frac{1}{p}D_1w_i^mh^I$ *lies in* $\mathcal{C}(S^{\langle i-1\rangle})$.

Proof. The first part in a) follows directly from (38); moreover under the hypothesis in a), 6.2 gives $\frac{1}{p}D_0w_i^mh^I\in C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$, but from its definition $\frac{1}{p}D_0w_i^mh^I\in\mathcal{C}(R^{\langle i\rangle})$, so that the second part in a) follows from (35). As for b), given $0<r\le m$ with $\nu\binom{m}{r}=0$, 4.8 implies $r\ge p^{\nu(m)}$, so that $m-r\le m-p^{\nu(m)}\le\nu(p^{\nu(m)}!)+\nu(I)+i-1\le\nu(r!)+\nu(I)+i-1$ (the second inequality being the hypothesis in b)), and it follows from (38) that $\frac{1}{p}(\!\binom{m}{r}\!)w_i^{m-r}[\vartheta_r]h^I$ lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$. Adding over all such r's we obtain

$$\frac{1}{p}D_1w_i^mh^I\in C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)}). \tag{39}$$

Thus (35) and (39) give $D_1w_i^mh^I\in\mathcal{C}(S^{\langle i\rangle})$ $(D_1w_i^mh^I\in\mathcal{C}(R^{\langle i\rangle})$ by 5.7). On the other hand, from a) we have $\frac{1}{p}dD_0w_i^mh^I\in\mathcal{C}(R^{\langle i\rangle})$, and since $\frac{1}{p}dD_0w_i^mh^I=$

$-\frac{1}{p}dD_1w_i^mh^I$, which by (39) lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$, we obtain from (35) $\frac{1}{p}dD_0w_i^mh^I\in\mathcal{C}(S^{\langle i\rangle})$. Finally the last assertion follows from the fact that $\frac{1}{p}D_1w_i^mh^I=D_1w_{i-1}^mh^I\in\mathcal{C}(R^{\langle i-1\rangle})$, together with (37) and (39). □

Lemma 6.4. *Let $\xi=\frac{1}{p}w_i^m\partial h^I$, $\zeta=\frac{1}{p}w_i^{m-(p-1)p^k}\partial g^{I'}$ and assume ξ is an element of type* 5.20 III *(so that $\nu(m)=k(I)=k$).*

a) *If ξ represents a class in $E_1(S^{\langle i\rangle})$ then so does ζ; moreover $\frac{1}{p}\overline{D}_0w_i^mh^I$ lies in $\mathcal{C}(S^{\langle i\rangle})$.*
b) *If ζ represents a class in $E_1(S^{\langle i\rangle})$ then both $\frac{1}{p}d\overline{D}_0w_i^mh^I$ and $\overline{D}_1w_i^mh^I$ lie in $\mathcal{C}(S^{\langle i\rangle})$, and if in addition $i\geq 1$ then $\frac{1}{p}\overline{D}_1w_i^mh^I$ lies in $\mathcal{C}(S^{\langle i-1\rangle})$.*

Proof. From its definition

$$\nu(I')-\nu(I)=\nu(p^{k+1}!)-\nu(p^k!)-1=p^k-1\geq 0, \tag{40}$$

so that the first part in a) follows from (38). On the other hand, the hypothesis in a) implies that the summand $\frac{1}{p}w_i^{m-(p-1)p^k}(D_{00}u_{k+1})h^{I-\Delta_k}$ in the definition of $\frac{1}{p}\overline{D}_0w_i^mh^I$ lies in $\mathcal{C}(S^{\langle i\rangle})$ and that

$$\frac{1}{p}w_i^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}\in C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$$

for $\lambda=1,\dots,p-1$ (in the first case the factor $\frac{1}{p}$ affects $D_{00}u_{k+1}$, which has Adams filtration 1 and is divisible by p in $C_\Gamma(\mathbb{Z}_{(p)};\mathbb{Z}_{(p)})$, and in the second case, $\frac{1}{p}$ affects $h^{I-\Delta_k}$, which, although of Adams filtration zero, is divisible by p in $C_\Gamma(\mathbb{Z}_{(p)};\mathbb{Z}_{(p)})$), so $\frac{1}{p}D_0'w_i^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}$ lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$ by 6.2, in $\mathcal{C}(R^{\langle i\rangle})$ by definition of D_0' and therefore in $\mathcal{C}(S^{\langle i\rangle})$ by (35). Thus $\frac{1}{p}\overline{D}_0w_i^mh^I\in\mathcal{C}(S^{\langle i\rangle})$ from its definition. The proof of b) is similar to that in 6.3, but this time we need to take into account the cancellations in (34). Since $d\overline{D}_0=-d\overline{D}_1$, we have to show that both $\frac{1}{p}d\overline{D}_1w_i^mh^I$ and $\overline{D}_1w_i^mh^I$ lie in $\mathcal{C}(S^{\langle i\rangle})$. Let $\overline{D}_1w_i^mh^I=A-B$, where $B=w_i^{m-(p-1)p^k}(D_{00}u_{k+1})h^{I-\Delta_k}$, and for $\lambda=1,\dots,p-1$ let $M_\lambda=w_i^{m-(\lambda-1)p^k}[\vartheta_{\lambda p^k}]h^{I-\Delta_k}$, so that

$$A=\frac{1}{a_m}\sum_{\lambda=1}^{p-1}(\!(\tbinom{m}{(\lambda-1)p^k})\!)(\!(\tbinom{\lambda p^k}{p^k})\!)^{-1}D_1'M_\lambda.$$

In view of (40) the hypothesis in b) is $-2+i+\nu(I)+p^k\geq m-(p-1)p^k$, which readily implies

$$\begin{gathered}\tfrac{1}{p}B,\ \ w_i^{m-(p-1)p^k}v_{k+1}h^{I-\Delta_k},\ \ \tfrac{1}{p}w_i^{m-(p-1)p^k}v_{k+1}\partial h^{I-\Delta_k}\in\mathcal{C}(S^{\langle i\rangle}),\\ \text{and}\qquad \tfrac{1}{p}w_i^{m-(p-1)p^k}v_{k+1}h^{I-\Delta_k}\in\mathcal{C}(S^{\langle i-1\rangle})\qquad\text{if }\ i\geq 1\end{gathered}\tag{41}$$

(the factor $\frac{1}{p}$ affects $D_{00}u_{k+1}$ in the first case, $\partial h^{I-\Delta_k}$ in the third and $w_i^{m-(p-1)p^k}=pw_{i-1}^{m-(p-1)p^k}$ in the fourth). In particular, we only need to show that A, $\frac{1}{p}dA\in\mathcal{C}(S^{\langle i\rangle})$ as well as $\frac{1}{p}A\in\mathcal{C}(S^{\langle i-1\rangle})$ for $i\geq 1$. From (34) the lowest weight filtration component of A is given by

$$\begin{aligned}\frac{1}{a_m}\sum_{\lambda=1}^{p-1} a_{\lambda m} w_i^{m-(p-1)p^k}&[\vartheta_{(p-\lambda)p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k}\\ &= \sum_{\lambda=1}^{p-1}\tfrac{1}{p}\left(\!\binom{p^{k+1}}{\lambda p^k}\!\right) w_i^{m-(p-1)p^k}[\vartheta_{(p-\lambda)p^k}|\vartheta_{\lambda p^k}]h^{I-\Delta_k}\\ &= -w_i^{m-(p-1)p^k}(D_{01}u_{k+1})h^{I-\Delta_k}\\ &= w_i^{m-(p-1)p^k}(D_{00}u_{k+1}-v_{k+1})h^{I-\Delta_k}\\ &= B - w_i^{m-(p-1)p^k}v_{k+1}h^{I-\Delta_k},\end{aligned}$$

which lies in $\mathcal{C}(S^{\langle i\rangle})$, and its $\frac{1}{p}$–multiple lies in $\mathcal{C}(S^{\langle i-1\rangle})$ in view of (41). Since this component has weight filtration $WF(h^{I-\Delta_k})+p^{k+1}$, we get, modulo weight filtration larger than this value,

$$\begin{aligned}\tfrac{1}{p}dA &= \tfrac{1}{p}d\left(B - w_i^{m-(p-1)p^k}v_{k+1}h^{I-\Delta_k}\right)\\ &= \tfrac{1}{p}dB - \tfrac{1}{p}w_i^{m-(p-1)p^k}\partial(v_{k+1}h^{I-\Delta_k})\\ &= \tfrac{1}{p}dB - \tfrac{1}{p}w_i^{m-(p-1)p^k}v_{k+1}\partial h^{I-\Delta_k}\end{aligned}$$

(recall that $v_{k+1}=\frac{1}{p}\partial u_{k+1}$ is a ∂–cycle of homology degree 2), where the last element in the equalities above lies in $\mathcal{C}(S^{\langle i\rangle})$ in view of (41). To complete the proof we have to show that the components in weight filtration larger than $WF(h^{I-\Delta_k})+p^{k+1}$ of both A and $\frac{1}{p}dA$ lie in $\mathcal{C}(S^{\langle i\rangle})$ and that the $\frac{1}{p}$–multiple of such components in A lie in $\mathcal{C}(S^{\langle i-1\rangle})$. In view of (37) it suffices to prove that the corresponding components of $\frac{1}{p}D'_1M_\lambda$ and $\frac{1}{p}dD'_1M_\lambda$ lie in $C_\Gamma(\ell_*(S^0);\mathbb{Z}_{(p)})$ (indeed, by their definitions D'_1M_λ, $\frac{1}{p}dD'_1M_\lambda=-\frac{1}{p}dD'_0M_\lambda\in\mathcal{C}(R^{\langle i\rangle})$, and since $w_i^n=pw_{i-1}^n$, $\frac{1}{p}D'_1M_\lambda\in\mathcal{C}(R^{\langle i-1\rangle})$).

The summands of weight filtration larger than $WF(h^{I-\Delta_k})+p^{k+1}$ in the definition of D'_1M_λ are, up to units, of the form $w_i^{m-(\lambda-1)p^k-r}[\vartheta_r|\vartheta_{\lambda p^k}]h^{I-\Delta_k}$ with $r>(p-\lambda)p^k$, and their $\frac{1}{p}$–multiples lie in $C_\Gamma(\ell_*(S^0);\mathbb{Z}_{(p)})$ since

$$\begin{aligned}-1+i+\nu(r!)+\nu((\lambda p^k)!)+\nu(I-\Delta_k)\geq&\\ \geq&\ -1+i+\nu(((p-\lambda)p^k)!)+\nu((\lambda p^k)!)+\nu(I)-\nu(p^k!)\\ =&\ -1+i+\nu(I)+(p-1)\nu(p^k!)\\ =&\ -2+i+\nu(I)+p^k\\ \geq&\ m-(p-1)p^k \qquad \text{(by the hypothesis in b))}\\ =&\ m-(\lambda-1)p^k-(p-\lambda)p^k\\ \geq&\ m-(\lambda-1)p^k-r.\end{aligned}$$

On the other hand a summand $\mathcal{S}$ of D'_1M_λ is, up to units, of the form

i) $w_i^{m-(\lambda-1)p^k-r}[\vartheta_r|\vartheta_{\lambda p^k}]h^{I-\Delta_k}$ with $\nu\binom{m-(\lambda-1)p^k}{r}=0$, or

ii) $w_i^{m-(\lambda-1)p^k}[\vartheta_{(\lambda-\mu)p^k}|\vartheta_{\mu p^k}]h^{I-\Delta_k}$ with $0<\mu<\lambda$, and we have to check that all summands in the definition of $\frac{1}{p}d\mathcal{S}$ of weight filtration larger than $WF(h^{I-\Delta_k})+p^{k+1}$ lie in $C_\Gamma(\ell_*(S^0);\mathbb{Z}_{(p)})$. This is verified by a direct calculation as above. In fact, since the power of p provided by $[\vartheta_n]$ to the coefficient in $\pi_*((\ell\wedge R^{\langle i\rangle})^{\langle 0\rangle})$ agrees with the corresponding power provided by $(\!(\binom{n}{s})\!)[\vartheta_{n-s}|\vartheta_s]$, we see that the above calculation works in all cases except for when $\mathcal{S}$ is of type i) above and the summand of $\frac{1}{p}d\mathcal{S}$ to be considered is of the form

$$\frac{1}{p}(\!(\tbinom{m-(\lambda-1)p^k-r}{s})\!)w_i^{m-(\lambda-1)p^k-r-s}[\vartheta_s|\vartheta_r|\vartheta_{\lambda p^k}]h^{I-\Delta_k}. \tag{42}$$

We complete the proof by giving full details for this situation. If $r>(p-\lambda)p^k$, the previous calculation certainly does work. So we assume $r\le(p-\lambda)p^k$. Since $\nu\left(\binom{m-(\lambda-1)p^k}{r}\right)=0$, 4.8 forces $r=\mu p^k$ with $0<\mu\le p-\lambda$. Moreover, our restriction on the weight filtration means $r+s>(p-\lambda)p^k$ or $s>(p-\lambda-\mu)p^k$, thus

$$\begin{aligned}
&-1+\nu\tbinom{m-(\lambda-1)p^k-r}{s}+i+\nu(s!)+\nu(r!)+\nu((\lambda p^k)!)+\nu(I-\Delta_k)\\
&\quad\ge\quad -1+i+\nu(s!)+\nu(r!)+\nu((\lambda p^k)!)+\nu(I-\Delta_k)\\
&\quad\ge\quad -1+i+\nu(((p-\lambda-\mu)p^k)!)+\nu((\mu p^k)!)+\nu((\lambda p^k)!)+\nu(I)-\nu(p^k!)\\
&\quad=\quad -1+i+\nu(I)+(p-1)\nu(p^k!)\\
&\quad=\quad -2+i+\nu(I)+p^k\\
&\quad\ge\quad m-(p-1)p^k \quad \text{(by the hypothesis in b))}\\
&\quad=\quad m-(\lambda-1)p^k-(p-\lambda)p^k\ge m-(\lambda-1)p^k-r-s,
\end{aligned}$$

which exhibits (42) as an element in $C_\Gamma(\ell_*(S^0);\mathbb{Z}_{(p)})$. □

Definition 6.5. Let $e_m=\max\{0,m-i\}$ and $e'_m=\max\{0,m+1-i\}$, so that $e'_m=e_m+\eta_m$, where $\eta_m=0$ if $m+1\le i$ and $\eta_m=1$ otherwise. By 6.1, $E_1(S^{\langle i\rangle})$ is $\mathbb{Z}_{(p)}$–free in homological degrees $s\le 1$, with basis as follows: for $s=0$ the elements $p^{e'_m}w_i^{m+1}$ $(m\ge -1)$ and for $s=1$ the elements $p^{e_m}w_i^m u_0$ $(m\ge 0)$. We will omit the index m in e_m, e'_m and η_m if no confusion arises.

Lemma 6.6. a) *For $j\ge\max\{e,\nu(m+1)\}$, $d(p^{j-\nu(m+1)}w_i^{m+1})$ lies in $\mathcal{C}(S^{\langle i\rangle})$ and agrees, up to units, with $p^jw_i^mu_0$ modulo weight filtration larger than* 1.

b) *For $e\le j<\nu(m+1)$, both $\frac{1}{p}w_i^{m+1-p^{j+1}}\partial u_{j+1}$ and A_j (as defined in 5.17 and replacing w^s by w_i^s in view of 5.2) lie in $\mathcal{C}(S^{\langle i\rangle})$.*

Proof. In a) it is clear that $p^{j-\nu(m+1)}w_i^{m+1}$ lies in $\mathcal{C}(R^{\langle i\rangle})$, and therefore so does its differential, which by 5.18 a) agrees, up to units and modulo weight filtration larger than 1, with $p^jw_i^mu_0$. Moreover the summands in the expression for $d(p^{j-\nu(m+1)}w_i^{m+1})$ are of the form $p^{j-\nu(m+1)}(\!(\binom{m+1}{r})\!)w_i^{m+1-r}[\vartheta_r]$ for $1\le r\le m+1$. Such a summand lies in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$ if and only if $j-\nu(m+1)+\nu\binom{m+1}{r}+i+\nu(r!)\ge m+1-r$, which, in view of the hypothesis $j\ge m-i$, holds provided

$$-\nu(m+1)+\nu\tbinom{m+1}{r}+\nu(r!)+r\ge 1. \tag{43}$$

If $r\ge\nu(m+1)$, (43) certainly holds; otherwise from 4.8 we obtain $\nu\binom{m+1}{r}=\nu(m+1)-\nu(r)$ and again (43) is clear. Thus a) follows from (35). For b) it suffices to prove $A_j\in\mathcal{C}(S^{\langle i\rangle})$ in view of 5.18 b). However the summands in the expression for A_j appear as summands of $d(p^{j-\nu(m+1)}w_i^{m+1})$, which as above lie in

$C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$. On the other hand, A_j lies in $\mathcal{C}(R^{\langle i\rangle})$ in view of 5.18 b) so that the result follows from (35). □

The next lemma is a straightforward consequence of 4.8.

Lemma 6.7. *For* $1\le t\le p^{j+1}$, $\nu((p^{j+1}-t)!)+\nu(t!)\ge\nu((p^{j+1}-1)!)$.

Lemma 6.8. *Let* $0\le j<\nu(m+1)$. *Then*

a) *if* $\frac{1}{p}w_i^{m+1-p^{j+1}}\partial u_{j+1}$ *represent a class in* $E_1(S^{\langle i\rangle})$ *then* $d(A_j)\in\mathcal{C}(S^{\langle i\rangle})$.

Assume in addition that $j\ge 1$; *then*

b) $d(A_j)-pd(A_{j-1})=\frac{1}{p^{\nu(m+1)-j}}d\left(\sum_{p^j\le r<p^{j+1}}(\!(\tbinom{m+1}{r})\!)w_i^{m+1-r}[\vartheta_r]\right)$, *and*

c) $d(A_{j-1})$, $d(A_j)$ *and* $\frac{1}{p^{\nu(m+1)-j}}(\!(\tbinom{m+1}{r})\!)w_i^{m+1-r}[\vartheta_r]$ $(p^j\le r<p^{j+1})$ *lie in* $\mathcal{C}(S^{\langle i\rangle})$ *provided* $\frac{1}{p}w_i^{m+1-p^j}\partial u_j$ *represents a class in* $E_1(S^{\langle i\rangle})$.

Proof. By 5.18 b), the hypothesis in a) means that the lowest weight filtration component in $d(A_j)$ (in the filtration p^{j+1}) does belong to $\mathcal{C}(S^{\langle i\rangle})$. Higher components are considered through each summand of $d(A_j)$:

$$\frac{1}{p^{\nu(m+1)-j}}d\left((\!(\tbinom{m+1}{r})\!)w_i^{m+1-r}[\vartheta_r]\right),\qquad 1\le r<p^{j+1}. \tag{44}$$

The summands of (44) in weight filtration larger than p^{j+1} are of the form

$$\frac{1}{p^{\nu(m+1)-j}}(\!(\tbinom{m+1}{r})\!)(\!(\tbinom{m+1-r}{t})\!)w_i^{m+1-r-t}[\vartheta_t|\vartheta_r]$$

with $1\le t\le m+1-r$ and $t+r>p^{j+1}$. Such elements lie in the cobar complex $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$, since

$$\begin{aligned}
&j-\nu(m+1)+\nu(\!(\tbinom{m+1}{r})\!)+\nu(\!(\tbinom{m+1-r}{t})\!)+i+\nu(t!)+\nu(r!)\\
&\quad\ge\ j-\nu(m+1)+\nu(\!(\tbinom{m+1}{r})\!)+i+\nu(t!)+\nu(r!)\\
&\quad=\ j-\nu(r)+i+\nu(t!)+\nu(r!)\quad\text{(by 4.8, since } r<p^{j+1})\\
&\quad=\ j+i+\nu(t!)+\nu((r-1)!)\\
&\quad\ge\ j+i+\nu(t!)+\nu((p^{j+1}-t)!)\quad\text{(since } t+r>p^{j+1})\\
&\quad\ge\ j+i+\nu((p^{j+1}-1)!)\quad\text{(by 6.7)}\\
&\quad=\ i+\nu(p^{j+1}!)-1\\
&\quad\ge\ m+1-p^{j+1}\quad\text{(by the hypothesis in a))}\\
&\quad\ge\ m+1-t-r
\end{aligned}$$

(if $t>p^{j+1}$ we skip the fifth term in these inequalities). Thus components in weight filtration larger than p^{j+1} in $d(A_j)$ lie in $C_\Gamma(\pi_*((\ell\wedge S^{\langle i\rangle})^{\langle 0\rangle});\mathbb{Z}_{(p)})$. Since they lie in $\mathcal{C}(R^{\langle i\rangle})$ (by 5.18 b)), they must also lie in $\mathcal{C}(S^{\langle i\rangle})$ in view of (35). This gives a). Part b) is an easy consequence of the formula

$$d(A_j)=-d(B_j)=-\frac{1}{p^{\nu(m+1)-j}}d\left(\sum_{r\ge p^{j+1}}(\!(\tbinom{m+1}{r})\!)w_i^{m+1-r}[\vartheta_r]\right).$$

Finally for $p^j\le r<p^{j+1}$, 4.8 gives $\nu\binom{m+1}{r}=\nu(m+1)-\nu(r)$, which together with $-1+i+\nu(p^j!)\ge m+1-p^j$ (the hypothesis in c)) implies $j-\nu(m+1)+$

$\nu\binom{m+1}{r} + i + \nu(r!) = j + i + \nu((r-1)!) \geq i + \nu(p^j!) > m + 1 - p^j \geq m + 1 - r$. Therefore $\frac{1}{p^{\nu(m+1)-j}}(\binom{m+1}{r})w_i^{m+1-r}[\vartheta_r]$ lies in $C_\Gamma(\pi_*((\ell \wedge S^{\langle i\rangle})^{\langle 0\rangle}); \mathbb{Z}_{(p)})$, and since $\nu\binom{m+1}{r} \geq \nu(m+1) - j$, it also lies in $\mathcal{C}(R^{\langle i\rangle})$. Thus c) follows from (35) and part a). □

We are now in position to give a full description of the WSS for $S^{\langle i\rangle}$. Recall from the remarks before 6.1 that $E_1(S^{\langle i\rangle}) \subseteq E_1(R^{\langle i\rangle})$.

Theorem 6.9. *Let y be an element in $E_1(S^{\langle i\rangle})$.*

a) *Suppose a differential $y \mapsto z$ holds in $E_r(R^{\langle i\rangle})$ (as described in 5.21). Then $z \in E_1(S^{\langle i\rangle})$ and the same differential holds in $E_r(S^{\langle i\rangle})$.*
b) *Suppose a differential $x \mapsto y$ holds in $E_r(R^{\langle i\rangle})$. Then y is a permanent cycle in $E_r(S^{\langle i\rangle})$.*

Proof. Pick representatives as in 5.22 for classes starting in $E_1(S^{\langle i\rangle})$. Lemmas 6.3, 6.4, 6.6 and 6.8 a) show that the analysis in 5.9, 5.16 and 5.19 of the differentials in 5.21 applies also for elements in $E_r(S^{\langle i\rangle})$, because the chosen representatives do lie in $\mathcal{C}(S^{\langle i\rangle})$. On the other hand Lemmas 6.3 b), 6.4 b), 6.6 a) and 6.8 a) show that any class in $E_1(S^{\langle i\rangle})$ that is the target of a differential in $E_r(R^{\langle i\rangle})$ is represented by a d–cycle in $\mathcal{C}(S^{\langle i\rangle})$ and therefore survives as a permanent cycle (perhaps as a δ_r–boundary) in the WSS for $S^{\langle i\rangle}$. Since this takes each class in $E_1(S^{\langle i\rangle})$ (except for the $\mathbb{Z}_{(p)}$ in weight filtration degree zero) into account once and only once, we get as in the proof of 5.21 that there is no room for the anomalous behavior described in 5.5 for any more differentials. □

Example 6.10. The following may help to understand the differentials in $E_r(S^{\langle i\rangle})$ in low homological degrees. We use the notation introduced in 6.5. The δ_1 differentials $p^{e'}w_i^{m+1} \mapsto p^{e'+\nu(m+1)}w_i^m u_0 = p^{\nu(m+1)+\eta}p^e w_i^m u_0$ yield $E_2^{\sigma,0,t}(S^{\langle i\rangle}) = 0$ for $(\sigma, t) \neq (0,0)$ and $E^{1,1,(m+1)q}(S^{\langle i\rangle}) = \mathbb{Z}/p^{\nu(m+1)+\eta}$ with generator represented by $p^e w_i^m u_0$. The classes with representatives $p^j w_i^m u_0$ for $e \leq j < \nu(m+1)$ produce higher differentials of type 5.21 d), whereas those with $\max\{e, \nu(m+1)\} \leq j < e' + \nu(m+1)$ produce nontrivial permanent cycles, so that $E_\infty^{1,1,(m+1)q}(S^{\langle i\rangle})$ is a cyclic group of order $p^{e'+\nu(m+1)-\max\{e,\nu(m+1)\}}$ ($E_1^{\sigma,1,t}(S^{\langle i\rangle}) = 0$ if $\sigma \neq 1$ or if $t \not\equiv 0 \pmod q$). On the other hand, the only classes of type 5.20 V that survive to a nontrivial element in $E_\infty(S^{\langle i\rangle})$ are those of the form $\frac{1}{p}w_i^{m+1-p^{j+1}}\partial u_{j+1} \in \mathcal{C}(S^{\langle i\rangle})$ with $0 \leq j < \min\{\nu(m+1), e\}$ (if $j \geq e$ such a class is killed by a differential of type 5.21 d) as above).

In the next result, the cohomology of $\mathcal{C}(S^{\langle i\rangle})$ is denoted by $H^*(i)$.

Corollary 6.11. a) *In homological degrees $s \leq 2$ the only possibly nontrivial cohomology groups of $\mathcal{C}(S^{\langle i\rangle})$ are given by*

- $H^{0,0}(i) = \mathbb{Z}_{(p)}$,
- $H^{1,(m+1)q}(i) = \mathbb{Z}/p^{a_m+1}$ *for $m \geq i$, and*
- $H^{2,(m+1)q}(i) = \mathbb{Z}/p^{\max\{0,\, a_m-\mu_m+1\}}$ *for $m \geq \max\{i+1, p-1\}$,*

where $a_m = \min\{m-i, \nu(m+1)\}$ and μ_m is the integral part of the logarithm in base p of $(p-1)(m+2-i)$.

b) *In homological degrees $s \geq 3$ the cohomology groups of $\mathcal{C}(S^{\langle i\rangle})$ are $\mathbb{F}_p$-free on generators represented by cycles of the form*

- $\frac{1}{p}dD_0w_i^mh^I$ *with* h^I *and* m *as in* 5.20 I *and* $-1+i+\nu(I)<m\le -1+i+\nu(I)+\nu(p^{\nu(m)+1}!)$
- $\frac{1}{p}d\overline{D}_0w_i^mh^I$ *with* h^I *and* m *as in* 5.20 III *and* $-1+i+\nu(I)<m\le -2+i+\nu(I)+p^{\nu(m)+1}$.

Proof. The examples in 6.10 give the result for $s\le 1$ (for dimensional reasons there are no nontrivial extensions in $H^1(i)$ other than those already happening in $E_\infty(S^{\langle i\rangle})$). For $s=2$ we have from 6.10 that the only elements in $E_1(S^{\langle i\rangle})$ surviving to nontrivial classes in $E_\infty(S^{\langle i\rangle})$ are those represented by elements of the form $\frac{1}{p}w_i^{m+1-p^{j+1}}\partial u_{j+1}$ satisfying

i) $0\le j<\nu(m+1)$,
ii) $j<e=\max\{0,m-i\}$, and
iii) $\frac{1}{p}w_i^{m+1-p^{j+1}}\partial u_{j+1}\in\mathcal{C}(S^{\langle i\rangle})$.

From i) and ii) we see that such classes exist only for $m\ge\max\{i+1,p-1\}$, in which case i) and ii) combine into the single condition $0\le j<a_m$. On the other hand, iii) means $-1+i+\nu(p^{j+1}!)\ge m+1-p^{j+1}$, which is easily seen to be equivalent to $p^{j+2}>(p-1)(m+2-i)$. Taking $\log_p$, this is $j\ge\mu_m-1$. Thus in total degree $(m+1)q$ and homological degree 2, $E_\infty(S^{\langle i\rangle})$ consists of $a_m-\mu_m+1$ copies of $\mathbb{F}_p$, all of which produce nontrivial extensions in $H^{2,(m+1)q}(i)$ in view of 5.22 (iii) and 6.8. This gives a). As for b), using the notation of 6.9 we see that the only surviving classes in $E_\infty(S^{\langle i\rangle})$ in homological degrees $s\ge 3$ come from classes $y\in E_1(S^{\langle i\rangle})$ involved in a differential of the form $x\mapsto y$ in the WSS for $R^{\langle i\rangle}$ and for which $x\notin E_1(S^{\langle i\rangle})$. Such classes y are represented by elements of types 5.20 II and IV or, in view of 5.22, by the cycles described in the statement of this corollary, where the upper bound given for m means $y\in E_1(S^{\langle i\rangle})$ whereas the lower bound means $x\notin E_1(S^{\langle i\rangle})$. Finally, for $s\ge 3$ there are no nontrivial extensions in $H^2(i)$ in view of the first assertions in 6.3 b) and 6.4 b). ∎

Remark 6.12. The bidegree (s,t) of each element in 6.11 b) is explicitly determined as follows: Let $s(I)$ and $\sigma(I)$ denote the homological and weight filtration degrees of the admissible monomial h^I. Thus

$$s(I)=\sum_{j\ge 0}(e_j+2m_j)$$

and

$$\sigma(I)=\sum_{j\ge 0}(e_j+m_j)p^j,$$

where $I=(e_0,e_1,m_1,e_2,m_2,\dots)$ $(m_0=0)$. Since the operators d, D_0 and $\overline{D}_0$ preserve total degree and increase the homological degree by one, we see that an element $\frac{1}{p}dD_0w_i^mh^I$ or $\frac{1}{p}d\overline{D}_0w_i^mh^I$ in 6.11 b) has total degree $t=q(m+\sigma(I))$ and homological degree $s=s(I)+2$.

Corollary 6.13. $H^{st}(i)=0$ *if* $s\ge 3$ *and* $t-s<(p^2-p-1)s+q(i-p+2)$.

Proof. Let s and t be the homological and total degrees, respectively, of an element $\frac{1}{p}dD_0w_i^mh^I$ or $\frac{1}{p}d\overline{D}_0w_i^mh^I$ in 6.11 b). From 6.12 and the lower bound for m in 6.11 b) we must have

$$t-s\geq q(i+\nu(I)+\sigma(I))-s(I)-2, \tag{45}$$

so that 5.3 gives

$$\begin{aligned} q(\nu(I)+\sigma(I)) &= q\Big(\sum_{j\geq 0}(e_j+m_j)\nu(p^j!)-\sum_{j\geq 0}m_j+\sum_{j\geq 0}(e_j+m_j)p^j\Big) \\ &= q\Big(\sum_{j\geq 0}e_j\nu(p^{j+1}!)+\sum_{j\geq 1}m_j(\nu(p^{j+1}!)-1)\Big) \\ &= \sum_{j\geq 0}2e_j(p^{j+1}-1)+\sum_{j\geq 1}2m_j(p^{j+1}-p). \end{aligned}$$

In the case of an element $\frac{1}{p}dD_0w_i^mh^I$, we know from 5.20 I that $\nu(m)<k(I)$, and in particular the first summation in the last equality runs over $j\geq 1$. In the case of an element $\frac{1}{p}d\overline{D}_0w_i^mh^I$, we have from 5.20 III that h^I starts as $u_ku_{k+d}\cdots$, so that the summation above starts as $2(p^{k+1}-1)+2(p^{k+d+1}-1)$, which is bounded below by $2(p-1)+2(p^2-1)$. In any case we get

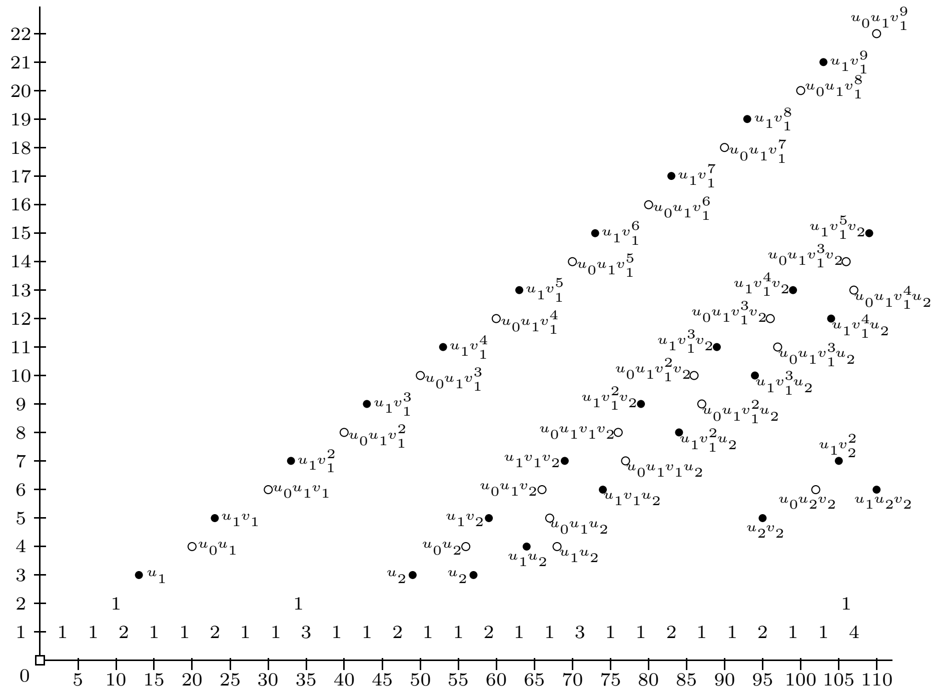
$$\sum_{j\geq 0}2e_j(p^{j+1}-1)+\sum_{j\geq 1}2m_j(p^{j+1}-p)\geq (p^2-p)\sum_{j\geq 0}(e_j+2m_j)+\nabla,$$

where $\nabla=2(p-1)+2(p^2-1)-2(p^2-p)=4(p-1)$, and (45) becomes

$$\begin{aligned} t-s &\geq qi+(p^2-p)s(I)+\nabla-s(I)-2 \\ &= (p^2-p-1)(s-2)+qi+\nabla-2=(p^2-p-1)s+D, \end{aligned}$$

where $D=qi+\nabla-2-2(p^2-p-1)=q(i-p+2)$. □

The corollary implies that in a chart with $(t-s,s)$ coordinates, the groups $H^{st}(i)$ lie on the right of the line with slope $(p^2-p-1)^{-1}$ and s-intersection at $q(p-i-2)(p^2-p-1)^{-1}$ (which is about 2 for $i=0$). By 6.12 every single element of $H^*(i)$ can be explicitly located in such a chart. As a way of example we sketch below the corresponding chart for $p=3$, $i=0$ and $t-s\leq 110$. The square at the origen represents a copy of $\mathbb{Z}_{(p)}$, a number n represents a cyclic group $\mathbb{Z}/p^n$ and a $\bullet$ (resp. $\circ$) represents a copy of $\mathbb{F}_p$ coming from an element $\frac{1}{p}dD_0w_i^mh^I$ (resp. $\frac{1}{p}d\overline{D}_0w_i^mh^I$), in which case we indicate the corresponding admissible monomial h^I giving rise to the element (the exponent m is determined by 6.11 b), except for the classes with $(t-s,s)=(49,3)$ and $(57,3)$, which have $m=4$ and $m=6$ respectively).



By 3.5 there is a short exact sequence of complexes

$$0 \to V(S^0) \to E_1(S^0;\ell) \to \mathcal{C}(S^0) \to 0 \tag{46}$$

which produces a long exact sequence involving $E_2(S^0;\ell)$—the second term in the ℓ–Adams spectral sequence for S^0. It is shown in [15] that the vanishing line given by 6.13 can be extended to a similar one in $E_2(S^0,\ell)$. The homotopical applications described in the introduction of the paper are based on a combination of such a vanishing line and the results in the next and final section.

7. Liftings through ℓ–resolutions

Recall from the beginning of section 6 that the Adams projections $S^{\langle i+1\rangle} \to S^{\langle i\rangle}$ produce monomorphisms of complexes $\mathcal{C}(S^{\langle i+1\rangle}) \to \mathcal{C}(S^{\langle i\rangle})$. The induced morphism in cohomology will be denoted by $A : H^*(i+1) \to H^*(i)$ (as in 6.11, $H^*(j)$ is shorthand for $H^*(\mathcal{C}(S^{\langle j\rangle}))$).

Lemma 7.1. *$A : H^s(i) \to H^s(i-1)$ is trivial if $s \geq 3$ and $i \geq 1$.*

Proof. Representatives in $\mathcal{C}(S^{\langle i\rangle})$ for the nontrivial classes in $H^s(i)$ are of the form $\frac{1}{p}dD_0w_i^m h^I = -\frac{1}{p}dD_1w_i^m h^I$ and $\frac{1}{p}d\overline{D}_0w_i^m h^I = -\frac{1}{p}d\overline{D}_1w_i^m h^I$ for suitably chosen m and I (6.11 b)). These elements map into boundaries in $\mathcal{C}(S^{\langle i-1\rangle})$ in view of the last assertions in 6.3 b) and 6.4 b). ☐

Lemma 7.2. *In homological degree $s = 2$ and total degree $t = (m+1)q$, the $\nu(m+1)$–fold composite*

$$A^{\nu(m+1)} : H^{st}(i) \xrightarrow{A} H^{st}(i-1) \xrightarrow{A} \cdots \xrightarrow{A} H^{st}(i-\nu(m+1))$$

is trivial.

Proof. Representatives in $\mathcal{C}(S^{\langle k\rangle})$ for the nontrivial classes in $H^{st}(k)$ (s and t as in the lemma) are given by the elements $e_j = \frac{1}{p} w_i^{m+1-p^{j+1}} \partial u_{j+1}$ for $\mu_m - 1 \leq j < \min\{m-i,\, \nu(m+1)\} = a_m$, which produce a cyclic group of order $p^{a_m - \mu_m + 1}$ (see 6.11 a) and its proof). The result follows since the Adams projection sends an element $e_j \in \mathcal{C}(S^{\langle k\rangle})$ into the p-multiple of $e_j \in \mathcal{C}(S^{\langle k-1\rangle})$, which produces in $H^*(k-1)$ the same homology class as e_{j+1} does. □

Definition 7.3. The spectrum J is the fiber of the map $\phi : \ell \to \Sigma^q \ell$ described in 2.4.

Since $\pi_*(\ell) = 0$ for $* \not\equiv 0 \pmod q$, the cofibration $J \to \ell \xrightarrow{\phi} \Sigma^q \ell$ induces short exact sequences

$$0 \to \pi_{(m+1)q}(\ell) \xrightarrow{\phi} \pi_{(m+1)q}(\Sigma^q \ell) \xrightarrow{\tau} \pi_{(m+1)q-1}(J) \to 0 \quad (m \geq 0), \tag{47}$$

and by (3) the group on the right is cyclic of order $\nu(m+1)+1$. On the other hand, by 3.6, the first component of the differential

$$\mathcal{C}^{0,\,(m+1)q}(S^0) \to \mathcal{C}^{1,\,(m+1)q}(S^0) = \bigoplus_{n>0} \pi_{(m+1)q}\left(\Sigma^{nq} \ell^{\langle \nu(n!)\rangle}\right) \tag{48}$$

is given by the map ϕ in (47). Thus the composition of the projection $\mathcal{C}^{1,\,(m+1)q}(S^0) \to \pi_{(m+1)q}(\Sigma^q \ell)$ and the map τ in (47) induce a well defined map $\lambda : H^{1,\,(m+1)q}(0) \to J_{(m+1)q-1}(S^0)$. Note from 6.11 that these two groups are isomorphic and in fact λ is an isomorphism, since according to 6.10, the representative for the generator in $H^{1,\,(m+1)q}(0)$ is $p^m w^m u_0$, which in terms of the sum in (48) corresponds to the generator in the first summand $\pi_{(m+1)q}(\Sigma^q \ell)$.

The topological consequences of the calculations in this paper depend on the next result, which is proven for the 2–primary situation in [11]. The proof goes without changes for odd primes. Recall that the Adams filtration of a homotopy class α is denoted by $AF(\alpha)$.

Lemma 7.4. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofibration of spectra and Z is a wedge of suspensions of $H\mathbb{F}_p$ and $\ell^{\langle k\rangle}$. Then for any class $\alpha \in \pi_n(Y)$ mapping trivially under g, there is a class $\beta \in \pi_n(X)$ such that $f_*(\beta) = \alpha$ and $AF(\beta) \geq AF(\alpha) - 1$.*

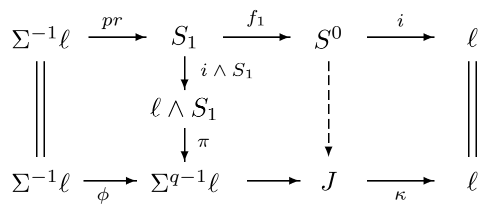
For dimensional reasons the unit $S^0 \xrightarrow{i} \ell$ maps trivially under ϕ, defining a unique class $S^0 \xrightarrow{j} J$ such that $i = \kappa \circ j$, where $J \xrightarrow{\kappa} \ell \xrightarrow{\phi} \Sigma^q \ell$ is the cofibration in 7.3.

Theorem 7.5. *Let $S^0 \xleftarrow{f_1} S_1 \xleftarrow{f_2} S_2 \xleftarrow{f_3} \cdots$ be the standard ℓ–resolution of S^0 (§2) and suppose given a homotopy class $\alpha_s \in \pi_n(S_s)$.*

a) *If $s = 0$ and $n > 0$, then there is a homotopy class $\alpha_1 \in \pi_n(S_1)$ such that $f_1(\alpha_1) = \alpha_0$ and $AF(\alpha_1) \geq AF(\alpha_0) - 1$.*
b) *If $s = 1$, $AF(\alpha_1) \geq 1$ and α_1 maps trivially under the composite $S_1 \xrightarrow{f_1} S^0 \xrightarrow{j} J$, then there is a homotopy class $\alpha_2 \in \pi_n(S_2)$ such that $f_1 \circ f_2(\alpha_2) = f_1(\alpha_1)$ and $AF(\alpha_2) \geq AF(\alpha_1) - 1$.*
c) *If $s = 2$ and $AF(\alpha_2) \geq \varepsilon_2$, then there is a homotopy class $\alpha_3 \in \pi_n(S_3)$ such that $f_2 \circ f_3(\alpha_3) = f_2(\alpha_2)$ and $AF(\alpha_3) \geq AF(\alpha_2) - \varepsilon_2$. Here $\varepsilon_2 = 1 + \nu(n+2)$ if $n+2 \equiv 0 \pmod q$, and $\varepsilon_2 = 1$ otherwise.*

d) *If* $s \geq 3$ *and* $AF(\alpha_s) \geq \varepsilon_s$, *then there is a homotopy class* $\alpha_{s+1} \in \pi_n(S_{s+1})$ *such that* $f_s \circ f_{s+1}(\alpha_{s+1}) = f_s(\alpha_s)$, *and* $AF(\alpha_{s+1}) \geq AF(\alpha_s) - \varepsilon_s$. *Here* $\varepsilon_s = 2$ *if* $n + s \equiv 0 \pmod q$, *and* $\varepsilon_s = 1$ *otherwise.*

Proof. Since the positive dimensional homotopy groups of S^0 are finite, α_0 maps trivially under $\pi_n(S^0) \xrightarrow{i} \pi_n(\ell)$, and so part a) follows immediately from 7.4. For b) consider the following diagram with cofiber rows:



Here π is the wedge projection given by 2.3 ($\Sigma S_1 = \overline{\ell}$); moreover, the square on the left commutes in view of 2.10. Thus the dashed line agrees with $j : S^0 \to J$. From (1) and (2), α_1 maps into a d_1–cycle $z \in E_1^{1,\,n+1}(S^0;\ell) = \pi_n(\ell \wedge S_1)$ in the first term of the ℓ–Adams spectral sequence. The current hypothesis together with the observations after 7.3 show that z maps in turn into a boundary in $\mathcal{C}(S^0)$ (see (46)). Therefore there is class $e \in E_1^{1,\,n+1}(S^0;\ell)$ carried by splitting $\mathbb{F}_p$ Eilenberg-Mac Lane spectra, and a class $w \in E_1^{0,\,n+1}(S^0;\ell) = \pi_{n+1}(\ell)$ such that

$$d_1 w = z - e. \tag{49}$$

As indicated in the diagram just before (35), every positive dimensional homotopy class in $\pi_*(\ell)$ must have positive classical Adams filtration. Thus $AF(d_1w) \geq AF(w) > 0$, and since $AF(z) \geq AF(\alpha_1) > 0$ it follows that $e = 0$. Recall from (2) that the differential d_1 in $E_1(S^0;\ell)$ is induced by the composite $\Sigma^{-1}\ell \xrightarrow{pr} S_1 \xrightarrow{i\wedge S_1} \ell \wedge S_1$. Letting $\alpha_1' = \alpha_1 - pr(w)$, (49) gives $(i \wedge S_1)(\alpha_1') = z - d_1w = e = 0$. Thus α_1' lifts through f_2, and since $f_1(\alpha_1') = f_1(\alpha) - f_1(pr(w)) = f_1(\alpha)$, part b) follows from 7.4 once we verify that $AF(\alpha_1') \geq AF(\alpha_1)$. If $n+1 \not\equiv 0 \pmod q$, w lies in a trivial group and we are done. Assume then $n + 1 = rq$. From 2.3, the r $^{\text{th}}$ wedge summand in $\ell \wedge S_1$ is $\Sigma^{rq-1}\ell^{\langle\nu(r!)\rangle}$, and from 3.6 the corresponding component of d_1w is $\phi_r(w)$. Therefore $\phi_r(w) = \rho_r(d_1w) = \rho_r(z)$, where $\rho_r : \ell \wedge S_1 \to \Sigma^{rq-1}\ell^{\langle\nu(r!)\rangle}$ is the wedge projection, and in particular

$$AF(\phi_r(w)) = AF(\rho_r(z)) \geq AF(z) \geq AF(\alpha_1). \tag{50}$$

From (3) we see that $\phi^r(v^r) = (k^r - 1)\cdots(k-1)\cdot 1$, and since $AF(v^r) = r$ and $\nu(k^j - 1) = \nu(j) + 1$, we get that ϕ^r increases the Adams filtration by $\nu(r!)$ in $\pi_{rq}(\ell)$. Thus 2.6 implies that ϕ_r preserves filtration in $\pi_{rq}(\ell)$, and by (50) $AF(w) = AF(\phi_r w) \geq AF(\alpha_1)$, which implies the required inequality $AF(\alpha_1') \geq AF(\alpha_1)$, completing the proof of part b).

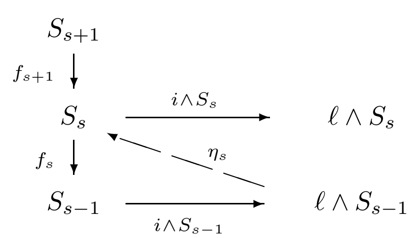
To analyze the ℓ–obstructions for liftings in the case $s \geq 2$, we use lemmas 7.1 and 7.2. Let $\alpha_s \in \pi_n(S_s)$ with $AF(\alpha_s) \geq \varepsilon_s$. As above, the element $z = (i \wedge S_s)(\alpha_s) \in \pi_n(\ell \wedge S_s) = E_1^{s,\,n+s}(S^0;\ell)$, having positive Adams filtration, lies in the $\mathbb{F}_p$ Eilenberg-Mac Lane–free portion of this group, that is, in $\pi_n((\ell \wedge S_s)^{\langle 0\rangle}) = \mathcal{C}^{s,\,n+s}(S^0)$. If $n + s \not\equiv 0 \pmod q$, such a group is trivial in view of (6) and (7), and in this case c) and d) follow directly from 7.4. For $n + s \equiv 0 \pmod q$ we note that

$$\begin{aligned}
\mathcal{C}^{s,\,n+s}(S^{\langle a\rangle}) &= \pi_{n+s}\left((\ell\wedge\overline{\ell}^{\,s}\wedge S^{\langle a\rangle})^{\langle 0\rangle}\right)\\
&= \pi_{n+s}\left(\left(\bigvee_{\overline{n}\in R_s}\Sigma^{\sigma(\overline{n})q}\ell^{\langle\nu(\overline{n}!)\rangle}\wedge S^{\langle a\rangle}\right)^{\langle 0\rangle}\right)\\
&= \pi_{n+s}\left(\left(\bigvee_{\overline{n}\in R_s}\Sigma^{\sigma(\overline{n})q}\ell^{\langle\nu(\overline{n}!)\rangle}\right)^{\langle a\rangle}\right)\\
&= \pi_n\left((\ell\wedge S_s)^{\langle a\rangle}\right)
\end{aligned}$$

in view of 3.4 and 2.3 (in that order). Thus in fact $z\in\mathcal{C}(S^{\langle a\rangle})$, where $a = AF(\alpha_s)\geq\varepsilon_s$. For $b = a-\varepsilon_s+1$, 7.1 and 7.2 imply that, as an element in $\mathcal{C}(S^{\langle b\rangle})$, z produces a trivial homology class; and, as in the case for $s=1$, this gives a class $w\in E_1^{s-1,\,n+s}(S^{\langle b\rangle};\ell)$ such that $z-d_1w$ maps trivially into $\mathcal{C}^{s,\,n+s}(S^{\langle b\rangle})$. As remarked in 2.1, these constructions are not functorial; however choices can be made so to have the following commutative diagram of complexes (see (46)), where vertical maps are "induced" by the Adams projection $S^{\langle b\rangle}\to S^0$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(S^{\langle b\rangle}) & \longrightarrow & E_1(S^{\langle b\rangle};\ell) & \longrightarrow & \mathcal{C}(S^{\langle b\rangle}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(S^0) & \longrightarrow & E_1(S^0;\ell) & \longrightarrow & \mathcal{C}(S^0) & \longrightarrow & 0
\end{array}$$

It follows that w and z, as elements in $E_1(S^0;\ell)$, satisfy the relation $z = d_1w$ (as elements in $E_1(S^0;\ell)$, w and z have Adams filtration at least b). We can now proceed as for the case $s=1$. In terms of the Adams resolution



the differential d_1 is given by the composite $(i\wedge S_s)\circ\eta_s$ (compare with (2)), so that the element $\alpha'_s = \alpha_s-\eta_s(w)$ maps trivially under $i\wedge S_s$ and has $AF(\alpha'_s)\geq\min\{AF(\alpha_s),AF(\eta_s(w))\}\geq b = a-\varepsilon_s+1$. The final conclusion follows from an application of 7.4 to the cofibration on top of the above diagram. □

Theorem 7.5 is the odd primary version of [11, Theorem 5.1]. We note however that the control on the Adams filtration for the lifting in b) is optimal.

References

[1] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Ann. of Math. (2)*, 72:20–104, 1960. MR **25:**4530
[2] J. F. Adams. On the groups $J(X)$, II. *Topology*, 3:137–171, 1965. MR **33:**6626

[3] J. F. Adams. Lectures in generalised cohomology. *Lecture Notes in Math.*, 99:1–138, 1969. MR **40:**4943

[4] J. F. Adams. The Kahn-Priddy theorem. *Proc. Cambridge Philos. Soc.*, 73:45–55, 1973. MR **46:**9976

[5] H. Cartan and S. Eilenberg. *Homological Algebra*. Princeton University Press, 1956. MR **17:**1040e

[6] G. Carlsson. On the stable splitting of $bo \wedge bo$ and torsion operations in connective K-theory. *Pacific J. Math.*, 87:283–297, 1980. MR **81j:**55003

[7] A. Clark. Homotopy commutativity and the Moore spectral sequence. *Pacific J. Math.*, 15:65–74, 1965. MR **31:**1679

[8] D. M. Davis. Odd primary *bo*-resolutions and K-theory localization. *Illinois J. Math.*, 30(1):79–100, 1986. MR **87g:**55026

[9] D. M. Davis, S. Gitler, and M. E. Mahowald. The stable geometric dimension of vector bundles over real projective spaces. *Trans. Amer. Math. Soc.*, 268(1):39–61, 1981. MR **83c:**55006

[10] D. M. Davis and M. E. Mahowald. Classification of the stable homotopy types of stunted real projective spaces. *Pacific J. Math.*, 125(2):335–345, 1986. MR **88a:**55008

[11] D. M. Davis and M. E. Mahowald. The image of the stable J-homomorphism. *Topology*, 28(1):39–58, 1989. MR **90c:**55013

[12] S. Eilenberg and J. C. Moore. Limits and spectral sequences. *Topology*, 1:1–23, 1962. MR **26:**6229

[13] J. Goldman and G. C. Rota. On the foundations of combinatorial theory IV: Finite vector spaces and Eulerian generating functions. *Studies in Appl. Math.*, 49:239–258, 1970. MR **42:**93

[14] J. González. Odd primary $BP\langle 1\rangle$-resolutions and classification of the stable summands of stunted lens spaces. *Trans. Amer. Math. Soc.* (to appear).

[15] J. González. A vanishing line in the $BP\langle 1\rangle$-Adams spectral sequence. Submitted to *Topology*.

[16] D. C. Johnson and W. S. Wilson. Projective dimension and Brown-Peterson homology. *Topology*, 12:327–353, 1973. MR **48:**12576

[17] R. M. Kane. Operations in connective K-theory. *Mem. Amer. Math. Soc.*, 34(254), 1981. MR **82m:**55025

[18] W. Lellmann. Operations and co-operations in odd-primary connective K-theory. *J. London Math. Soc. (2)*, 29(3):562–576, 1984. MR **88a:**55020

[19] W. Lellmann and M. E. Mahowald. The *bo*-Adams spectral sequence. *Trans. Amer. Math. Soc.*, 300(2):593–623, 1987. MR **88c:**55023

[20] M. E. Mahowald. *bo*-resolutions. *Pacific J. Math.*, 92(2):365–383, 1981. MR **82m:**55017

[21] M. E. Mahowald. The image of J in the EHP sequence. *Ann. of Math. (2)*, 116(1):65–112, 1982; errata, *Ann. of Math.* (2), 120:399–400, 1984. MR **83i:**55019; MR **86d:**55018

[22] M. E. Mahowald and R. J. Milgram. Operations which detect Sq^4 in connective K-theory and their applications. *Quart. J. Math. Oxford Ser. (2)*, 27(108):415–432, 1976. MR **55:**6429

[23] H. R. Margolis. Eilenberg-Mac Lane spectra. *Proc. Amer. Math. Soc.*, 43:409–415, 1974. MR **49:**6239

[24] H. R. Margolis. *Spectra and the Steenrod Algebra: Modules over the Steenrod Algebra and the Stable Homotopy Category*. North-Holland, 1983. MR **86j:**55001

[25] R. J. Milgram. The Steenrod algebra and its dual for connective K-theory. In *Reunión sobre Teoría de Homotopía*, pages 127–158. Sociedad Matemática Mexicana, 1975. MR **86b:**55001

[26] H. R. Miller. On relations between Adams spectral sequences with an application to the stable homotopy of a Moore space. *J. Pure Appl. Algebra*, 20(3):287–312, 1981. MR **82f:**55029

[27] J. C. Moore. Algèbre homologique et homologie des espaces classifiants. *Séminaire Henri Cartan*, exposé 7, 1959/60. MR **28:**1092

[28] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, 1986. MR **87j:**55003

[29] W. J. Wong. Powers of a prime dividing binomial coefficients. *Amer. Math. Monthly*, 96(6):513–517, 1989. MR **90d:**05011

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